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Perspectives on Pfaffians of heterotic world-sheet instantons

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Abstract: To fix the bundle moduli of a heterotic compactification one has to understand the Pfaffian one-loop prefactor of the classical instanton contribution. For compactifications on elliptically fibered Calabi-Yau spaces X this can be made explicit for spectral bundles and world-sheet instantons supported on rational base curves b: one can express the Pfaffian in a closed algebraic form as a polynomial, or it may be understood as a θ-function expression. We elucidate the connection between these two points of view via the respective perception of the relevant spectral curve, related to its extrinsic geometry in the ambient space (the elliptic surface in X over b) or to its intrinsic geometry as abstract Riemann surface. We identify, within a conceptual description, general vanishing loci of the Pfaffian, and derive bounds on the vanishing order, relevant to solutions of the equations $W = dW = 0$ for the superpotential.

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1 Introduction

In a heterotic string compactification on a Calabi-Yau space X with bundle V, giving $N = 1$ supersymmetry in the effective 4D theory, world-sheet instantons can generate an effective superpotential for the moduli, thus potentially partially lifting the classical moduli space.[1] This involves a moduli-dependent one-loop functional determinant. We will study world-sheet instantons, wrapping (once) smooth holomorphic curves b in X. As contributions to the superpotential W arise only at tree-level in string perturbation theory, a contributing b has genus zero. As just two fermion zero-modes on b are needed to generate a superpotential, one has, conjecturally,[2] as further necessary condition that b is isolated (otherwise additional fermion zero-modes on P arise).[3] So we assume throughout that b is a smooth, isolated rational curve. Actually we take X elliptically fibered $\pi : X \to B$ (with section σ and fibre F) and b a curve in the base B (embedded via σ). For $\mathrm{SU}(n)$ bundles V which arise (for all details on the notions and notations of the spectral cover method cf. appendix A.3) as $V = p_*(p_C^* L \otimes \mathcal{P})$ from a spectral cover surface C (an n-fold ramified cover of B) endowed with a line bundle L (say, arising via restriction from a line bundle $\underline{L}$ on X) one has a cover curve c of b and a line bundle $l = \underline{L}|_{\mathcal{E}}$ on $\mathcal{E} = \pi^{-1}(b)$ with $V|_b \cong \pi_{c*} l|_c$. Let us denote $l(-F) = l \otimes \mathcal{O}_{\mathcal{E}}(-F)$ by $\widetilde{\mathcal{L}}$, and let $\mathcal{L} := \widetilde{\mathcal{L}}|_c$ and $\mathcal{L}_{c'} := \widetilde{\mathcal{L}}|_{c'}$ for $c' \in |c|$ (the system of linear equivalent divisors) or $\mathcal{L}_{c_t}$ for $t \in \mathcal{M}_{\mathcal{E}}(c)$.

To understand on a computational level the moduli dependence of the Pfaffian prefactor $Pfaff$ in the superpotential W_b caused by a world-sheet instanton supported on b (cf. section 2) we note that, from different perspectives, two equivalent expressions arise

- a polynomial expression in algebraic moduli (parameters describing motion in $|c|$)
- a θ-function expression in transcendental moduli

The two ensuing expressions for $Pfaff$ are of course related with each other.

Recall that a non-trivial contribution W_b of b to the world-sheet instanton superpotential occurs precisely for $H^0(b, V|_b \otimes \mathcal{O}_b(-1)) = 0$. So the vanishing locus of $Pfaff$ is given by the vanishing locus of an expression which controlles the non-triviality of $H^0(c, \mathcal{L}_{c_t})$ as t varies over the relevant moduli space. (This leads, via arguments of holomorphy and a consideration of the possible power occurring, to an identification of the Pfaffian with such a 'controlling expression', typically a certain determinant, cf. below).

[1] We treat a single instanton contribution; the sum might vanish non-trivially in some situations.

[2] It is not clear that additional fermion zero-modes could not be lifted by higher-order interactions.

[3] Instead of generating a superpotential world-sheet instantons may also deform the complex structure of the classical moduli space; this possibility is realized if b moves in a family [6].

Now let us come back to the expressions for the Pfaffian. The first expression, giving the polynomial $\det \iota_1$, arises via the *extrinsic* algebraic exact[4] sequence

$$0 \longrightarrow H^0(c, \mathcal{L}) \xrightarrow{\delta} H^1(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) \xrightarrow{\iota_1} H^1(\mathcal{E}, \widetilde{\mathcal{L}}) \longrightarrow H^1(c, \mathcal{L}) \longrightarrow 0 \tag{1.1}$$

with ι_1 induced from multiplication with a moduli-dependent element $\tilde{\iota} \in H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$. On the other hand one has also the *intrinsic* transcendental exact sequence

$$0 \longrightarrow H^0(c, \mathcal{L}) \longrightarrow \mathcal{C}^\infty(c, \mathcal{L}) \xrightarrow{\bar{\partial}} \mathcal{C}^\infty(c, \mathcal{L} \otimes \Omega_c^{0,1}) \longrightarrow H^1(c, \mathcal{L}) \longrightarrow 0 \tag{1.2}$$

where the determinant of the $\bar{\partial}$ operator gives rise to the θ-function expression. We connect these two points of view by showing how the θ-function expression arises also in the algebraic framework, cf. (6.4). Further the sequences are interwoven with each other

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & H^0(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) & \rightarrow & H^0(\mathcal{E}, \widetilde{\mathcal{L}}) & \rightarrow & H^0(c, \mathcal{L}) & \xrightarrow{\delta} \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{C}^\infty(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) & \rightarrow & \mathcal{C}^\infty(\mathcal{E}, \widetilde{\mathcal{L}}) & \rightarrow & \mathcal{C}^\infty(c, \mathcal{L}) & \\
 & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & \\
0 \rightarrow & \mathcal{C}^\infty(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c) \otimes \Omega_{\mathcal{E}}^{0,1}) & \rightarrow & \mathcal{C}^\infty(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \Omega_{\mathcal{E}}^{0,1}) & \rightarrow & \mathcal{C}^\infty(c, \mathcal{L} \otimes \Omega_c^{0,1}) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow{\delta} & H^1(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) & \xrightarrow{\iota_1} & H^1(\mathcal{E}, \widetilde{\mathcal{L}}) & \rightarrow & H^1(c, \mathcal{L}) & \rightarrow 0 \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array} \tag{1.3}$$

Here the first two H^0-groups vanish if b is able to contribute to the superpotential at all, cf. (6.1), a necessary condition we assume to be fulfilled throughout.

The algebraic sequence arises *horizontally*, the transcendental one *vertically*; when considering the moduli space fibration there will be a second reason for these denotations.

A θ-function representation for $Pfaff$ arises generally from the transcendental sequence (1.2), cf. section 2.1. After recalling how the algebraic expression arises from (1.1) we show how a θ-function arises also directly in the algebraic approach, cf. (6.4).

Because the chiral Dirac operator is closely related to the $\bar{\partial}$-operator one can approach all statements within the algebraic geometric category. In such a framework the (zero) divisor given by the vanishing locus of $Pfaff$ in the moduli space will be crucial. We continue our study [9] of vanishing loci of $Pfaff$. The codimension zero loci are given by those components of the reducible moduli space for which the purely topological criterion for non-contribution (of the world-sheet instanton supported on b) to the superpotential applies. In the next codimension the mentioned vanishing divisor occurs; this divisor will in general be neither irreducible nor will its components be reduced: for corresponding examples of $Pfaff = fg$ or $Pfaff = fg^k$ cf. [9, 10] and section 4.6.3. In these examples

[4] We made the technical assumption $\lambda > 1/2$ on the spectral twist parameter λ and assume the necessary condition $h^0(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) = h^0(\mathcal{E}, \widetilde{\mathcal{L}})$ for contribution of b to W to be fulfilled (section 4.2 [9]).

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we could give explicit factors of $Pfaff$ from structural reasons [9]. More generally the corresponding loci arising from such structural arguments have higher codimension; they are specific subvarieties of the vanishing divisor which have an explicit description, cf. (4.34); we will give the corresponding nested hierarchy of loci, cf. section 4.3 and 4.4; it would be interesting to compare these algebraic geometric descriptions with corresponding facts on the side of the explicit transcendental θ-function.

Furthermore the mentioned loci are typically of higher multiplicity. This is (besides the θ-function) the other topic considered here which goes beyond the considerations in [9]. Although we will not touch the question of summing up the world-sheet instanton superpotential from its different contributions (supported on different curves) we consider the question of solutions to $W = dW = 0$ on the level of an individual instanton contribution: what is meant here comes down to studying the multiplicity k (later denoted by k') with which a factor f occurs in $W = f^k g$, in particular whether it is ≥ 2. We will study the question of multiplicity in some generality before we apply this to the two main examples of [10] (which were also treated in [9]) and the generalisation given in [9], cf. section 4.5 and 4.6, deriving, in particular, conceptually that multiplicities ≥ 2 occur there. This supplements and completes our conceptual treatment of the factor f^k in [9].

In section 2 we recall the definition of the Pfaffian prefactor and some subtleties connected with it. We also point to the interpretation of $Pfaff$ as a section of the square root of the determinant line bundle for a family of complex chiral Dirac operators. This usually occurs for a family of varying curves (with associated Dirac or $\bar{\partial}$ operators); in our case the support curve b of the world-sheet instanton stays fixed and the bundle over it varies; for our case of a base curve $b \subset B$ for spectral cover bundles over an elliptically fibered Calabi-Yau space X this movement in bundle moduli space comes down again to the variation of the spectral cover curve c which lies in the elliptic surface $\mathcal{E} = \pi^{-1}(b)$ over b. In this connection we also point to the general theta-function expression arising in such a set-up. In section 3, 4 and *5* we study in detail the bundle moduli spaces over X, $\mathcal{E}$ and b. We will assume that there are no continuous moduli for line bundles over the spectral cover surface $C \subset X$, lying over B (in the moduli space fibration they correspond to a fibre direction, cf. [8]). Then the bundle moduli are given, up to a discrete parameter $\lambda \in \frac{1}{2}\mathbf{Z}$ (still a fibre direction in moduli space), by the movements of C in X (horizontal base direction in moduli space). By restriction corresponding statements ensue for the case of $c \subset \mathcal{E}$, lying over b. Here, however, the moduli space for spectral bundles which are defined a priori over $\mathcal{E}$ (not arising by restriction from X) has also continuous fibre direction moduli as here the Jacobian of c enters the story. In section 4.3–4.6 we give the nested hierarchy of vanishing loci, cf. (4.34), discuss the question of multiplicity and put examples into this context in section 4.6.3. In section 6 we demonstrate how the theta-function arises directly in the algebraic geometric approach and compare the moduli space description of the restricted bundle $V|_{\mathcal{E}}$ and the corresponding universal Jacobian fibration over the moduli space $\mathcal{M}_g$ of curves where the general transcendental theta-function lives naturally; we conclude with remarks on multiplicities. The appendices collect various technical material related to the spectral cover construction (the algebraic approach, appendix A) and the Riemann theta function (the transcendental approach, appendix B).

1.1 Overview and summary

As the issue in question - the vanishing behaviour of the Pfaffian prefactor (of a world-sheet instanton superpotential) in dependence on the vector bundle moduli - necessarily uses a heavy amount of algebraic-geometric notions it may be useful to provide here also a nontechnical overview of the more detailled investigations which follow in the later sections (herein we allow ourselves to give an only approximate description of various issues whose potential subtleties are dealt with in the main text).

Clearly one first has to understand the moduli in question. As the bundle decomposes fibrewise in a sum of line bundles, each of which can be represented as $\mathcal{O}_F(q_i - p_0)$ by a fibre point $q_i \in F$, the bundle corresponds to a n-fold cover surface C of B, or a corresponding curve c over b. If $s = 0$ is an equation for c in $\mathcal{E} = \pi^{-1}(b)$ then it is the variation of these zerodivisors (s) in the linear system $|c| = \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(c))$ which gives the variation of the bundle V (concretely this comes down to having different polynomial coefficients a_i in the affine expression $s = a_0 + a_2 x + a_3 y + \dots$).

In addition the construction can be twisted by a line bundle $\mathcal{L} = \mathcal{O}_\mathcal{E}(D)$ on c of degree $g_c - 1$; so one has a representation

$$\mathcal{L} = K_c^{1/2} \otimes \mathcal{F} \tag{1.4}$$

where $\mathcal{F}$ is a flat line bundle on c. Actually there is a universal expression for $\mathcal{L}$ as coming from a restriction to c of a line bundle $\widetilde{\mathcal{L}}$ on $\mathcal{E}$ (here occur some subtleties concerning the question of integrality versus half-integrality of cohomology classes for different parities of parameters involved); one has $\mathcal{F} = \Lambda|_c^\lambda$ with $\Lambda = \mathcal{O}_\mathcal{E}(ns - (r-n)F)$ and λ a half-integer.

The main contribution criterion states that $Pfaff$ vanishes just if $V|_b \otimes \mathcal{O}_b(-1)$ has nontrivial sections

$$Pfaff(t) = 0 \Longleftrightarrow \Gamma\Big(b, V|_b \otimes \mathcal{O}_b(-1)\Big) \neq 0 \tag{1.5}$$

what translates for spectral bundles to the question whether $\mathcal{L}$ has nontrivial sections. So, for example, one finds that on the locus Σ in the moduli space $\mathcal{M}_\mathcal{E}(c)$ (given by the different concrete curves c_t in $|c|$) where $\Lambda|_{c_t}$ becomes trivial the divisor D of $\mathcal{L}$ becomes effective and so $Pfaff$ vanishes (various refinements of this locus Σ will be investigated)

$$\Sigma \subset (Pfaff) \tag{1.6}$$

The description of this locus is on the one hand satisfying as it is defined purely structurally. On the other hand one would also like to connect this with explicit descriptions. The latter means an explicit expression given directly in the moduli (the coefficients of the a_i). This is accomplished by the definition of the locus $\mathcal{R}$ where all the resultants $R_i^{(j)} = Res(a_i, a_n^{(j)})$ vanish (with $i = 2, \dots, n-1$ and $a_n = \prod_j a_n^{(j)}$) which will be shown to be a sublocus of Σ

$$\mathcal{R} \subset \Sigma \tag{1.7}$$

For our main case of interest, the case of SU(3) bundles (where also all the main examples occur), one gets even equality between these subloci in moduli space, giving a precise *explicit* description of the locus Σ in the vanishing divisor $(Pfaff)$ of $Pfaff$

$$\mathcal{R} = \Sigma \qquad\qquad \text{for SU(3) bundles} \tag{1.8}$$

We can be even more concrete: for SU(3) bundles (as for all SU(n) bundles with n odd) the half-integral parameter λ in the spectral construction has actually to be strictly half-integral. As we have furthermore the standing technical assumption $\lambda > 1/2$ (cases with $\lambda < -1/2$ can be reduced to this situation) actually the first interesting case is $\lambda = 3/2$ which is also a major case with regard to the concrete applications we have in mind (in view of some concretely known examples). In this case a whole component (f) of $(Pfaff)$ could be described (i.e. a sublocus of codimension zero; this comes from a relation $f|Pfaff$) which also contains the locus $\mathcal{R} = \Sigma$.

When one combines this with certain insights one can gain structurally about the multiplicity k with which f occurs in $Pfaff = f^k g$ one can recover the decisive assertions in most of the main examples known of the factorisation phenomenon for $Pfaff$ as will be described in section 4.6.3.

Finally the whole discussion can be described also from the internal perspective of the algebraic curve c. Then the different moduli are related to the moduli of the abstract Riemann surface c and the flat line bundle $\mathcal{F}_t$ gives for each concrete curve c_t a point in the Jacobian $Jac(c_t)$. As both vanishing divisors in the appropriate moduli spaces are related to the effectivity of a certain divisor (or the existence of a nontrivial section) it turns out that $(Pfaff)$ is most closely related to the theta divisor Θ. This is nice because it shows how the transcendental theta function, over a certain locus, can be expressed - if one switches from the transcendental period variables suitable to the universal Jacobian fibration over the moduli space of curves to the algebraic polynomial parameters described earlier - as a finite determinant (the connection can be deepened for the multiplicities).

2 The Pfaffian prefactor

The first level of question is whether a specific world-sheet instanton does make at all a contribution to the superpotential or not. In the positive case one wants of course to have more specific information about the prefactor f to the exponential instanton contribution

$$W_b = f \cdot \exp\left(i \int_b \tilde{J}\right) \tag{2.1}$$

Here $\tilde{J} = B + iJ$ is the Kähler class with area$(b) = 2\pi\alpha' \int_b J$. The prefactor f stems from the one-loop integral over quantum fluctuations around the classical instanton solution giving the exponential term.[5] We recall here some generalities relevant to f.

[5] We will not give a full elucidation of the instanton contribution which is somewhat subtle as the different naive factors of the moduli space are not decoupled. The pfaffian is a section of a line bundle which can be non-trivial if V leads to a non-vanishing fivebrane class $W_5 \neq 0$; however, even for $W_5 = 0$, there is by the

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With the kinetic Lagrangian $S_\psi = \int_b tr\,\bar\psi\, D_- \psi\; d\sigma d\tau$ the fermionic partition function gives $\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{-S_\psi} \sim$ det $\mathcal{D}_F$ ($\mathcal{D}_{B/F}$ the kinetic term for bosonic/fermionic fluctuations) in Euclidean space with Weyl-fermions. $\mathcal{D}_F$ decomposes on $S\otimes V|_b$ according to the chirality decomposition of the spinor bundle $S = S_+ \oplus S_-$ on b: in the left-moving sector one has the chiral Dirac operator $D_- : S_- \otimes V|_b \longrightarrow S_+ \otimes V|_b$ (with gauge connection A) and on the other hand $i\partial_+ : S^+ \otimes V|_b \longrightarrow S^- \otimes V|_b$ (without A).

In the decomposition $\det \mathcal{D}_F = |\det \mathcal{D}_F|\, e^{i\phi}$ the absolute value is evaluated (with a universal positive proportionality constant) via $|\det \mathcal{D}_F|^2 = \det\left(\mathcal{D}_F\,\mathcal{D}_F^\dagger\right) \sim \det \mathcal{D} = \det\left(D_- D_-^\dagger\right)$ which is a gauge-invariant quantity (here $D_+ = D_-^\dagger$ replaces the $i\partial_+$ above, leading to $\mathcal{D}$ replacing the $\mathcal{D}_F$ above); this leads[6] to $|\det \mathcal{D}_F| \sim \sqrt{\det \mathcal{D}}$.

Converting the treatment of Weyl-fermions in Euclidean space to the physical case of Majorana-Weyl fermions in Minkowski signature halves the number of degrees of freedom and leads to taking the square root $\sqrt{\det \mathcal{D}_F} = \mathrm{Pfaff}(\mathcal{D}_F)$. Then the prefactor f in (2.1) can be evaluated as [4] (the prime stands for omission of zero-modes)

$$f = \frac{\mathrm{Pfaff}'\,(\mathcal{D}_F)}{\sqrt{\det'\,\mathcal{D}_B}} = \frac{\mathrm{Pfaff}\,(\bar\partial_{V|_b(-1)})}{(\det\,\bar\partial_{\mathcal{O}(-1)})^2(\det'\,\bar\partial_{\mathcal{O}})^2} \tag{2.2}$$

The world-sheet instanton computation is often considered in the physical gauge. The bosons, describing fluctuations of b in its ten-dimensional ambient space, are $\mathcal{O}^2 \oplus N$-valued, using a complexification of four-dimensional extrinsic space and the normal bundle N of b in X. For a smooth rational curve which is isolated one gets $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ for the normal bundle. Interpreting the real eight-dimensional normal bundle as complex four-dimensional one may write $\sqrt{\det \mathcal{D}_B} = \det \mathcal{D}_B'$.

To describe the world-volume fermions consider the spinor bundle $S = S_-(b) \oplus S_+(b) = K_b^{1/2} \oplus K_b^{-1/2}$ on b (with its left/right-movers decomposition and after chosing an appropriate complex structure) with respective kinetic operators $\bar\partial$ and ∂. The right-moving fermions are sections of $S_+(b) \otimes S_+(\mathcal{O}^2 \oplus N)$ where $S_+(\mathcal{O}^2 \oplus N) = \Big(S_+(\mathcal{O}^2) \otimes S_+(N)\Big) \oplus \Big(S_-(\mathcal{O}^2) \otimes S_-(N)\Big)$ is the positive chirality spin bundle associated to N.

Because of unbroken supersymmetry in the instanton field there is a full cancellation in the right-moving sector between fermions contributing to $Pfaff'(\mathcal{D}_F)$ and bosons contributing to $\det' \mathcal{D}_B$. The rewriting in (2.2) in the remaining left-moving sector (retaining just $\bar\partial$ operators) results from the identification

$$\text{left-moving fermions} \cong \Gamma\Big(b, S_-(b) \otimes V|_b\Big) \tag{2.3}$$

D_- becomes $(i)\bar\partial$ (for a suitable choice of complex structure) and the left-handed spin bundle $S_-(b)$ becomes $\mathcal{O}_b(-1)$ (we assume structure group $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$ with $n \leq 8$).

lack of a canonical trivialization no well-defined phase of the pfaffian [4] (its absolute value can be defined by zeta-function regularization); the phase of the classical exponential (Kähler) factor has a non-trivial interference with the bundle moduli [5], yet the total phase in (2.1) is well-defined; another subtlety is that the stability notion for the bundle depends on the Kähler class employed.

[6] In 2, or more generally $8k+2$, dimensions the determinant line bundle of the complex chiral Dirac operator admits a canonical square root and $\sqrt{\det \mathcal{D}}$ is a section.

The contribution criterion states that $W_b \neq 0$ just if $\bar{\partial}|_{V|_b(-1)}$ has a zero kernel $H^0(b, V|_b(-1))$. If not $H^0(b, V|_b(-1)) = 0$ everywhere in moduli space $\mathcal{M}$, i.e. if $Pfaff \not\equiv 0$, it will give in our spectral case a divisor defined by $\det \iota_1 = 0$ in $\mathcal{M}$, where $Pfaff = 0$, such that $Pfaff = (\det \iota_1)^m$ up to a constant (actually $m = 1$ [10]).

2.1 The Pfaffian as a section of a line bundle

The Pfaffian is a section of the canonically existing square root of the determinant line bundle for a family of complex chiral Dirac operators. A holomorphic family of complex curves X_m (with $m \in \mathcal{M}$) gives a family of bundles

$$\begin{array}{l} \mathcal{X} \\ \downarrow \; S_-(X_m) \otimes V_X \\ \mathcal{M} \end{array} \tag{2.4}$$

and thereby a family $(\bar{\partial}_m)_{m\in\mathcal{M}}$ of $\bar{\partial}$-operators on $\mathcal{C}^\infty(X_m, V)$. From the sequence

$$0 \longrightarrow H^0(X_m, V) \longrightarrow \mathcal{C}^\infty(X_m, V) \xrightarrow{\bar{\partial}_m} \mathcal{C}^\infty(X_m, V \otimes \Omega^{0,1}_{X_m}) \longrightarrow H^1(X_m, V) \longrightarrow 0 \tag{2.5}$$

one gets a family of one-dimensional vector spaces

$$\mathrm{DET}(\bar{\partial}_m) \;\; = \;\; \Lambda^{h_0} H^0(X_m, V) \otimes \Lambda^{h_1} H^1(X_m, V)^* \tag{2.6}$$

These complex lines fit together holomorphically over $\mathcal{M}$ to give the determinant line bundle $\mathrm{DET}(\bar{\partial})$ (endowed with a suitable norm) over $\mathcal{M}$ with fibre $\mathrm{DET}(\bar{\partial}_m)$.

The varying family of $\bar{\partial}$ operators may also be obtained if the relevant bundle changes instead of the curve. As in the spectral cover approach the bundle is encoded (essentially) again by a spectral cover *curve* this will lead back in the end again to a curve c, varying over its moduli space (cf. the space $\mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$ below in section 4).

Remark. The theta function can be understood as the determinant of an appropriate $\bar{\partial}$ operator: choosing first a spin structure, i.e. a theta characteristic $K_c^{1/2}$, allows an identification $\phi : Pic_{g-1}(c) \longrightarrow Jac(c)$ via the relations (where $\mathrm{Div}^{eff}_{g-1}(c) = Sym^{g-1}\, c$)

$$\begin{array}{ccc} \mathrm{Div}^{eff}_{g-1}(c) \rightarrow \Big(\mathrm{Div}^{eff}_{g-1}(c)/\sim\Big) \hookrightarrow \Big(\mathrm{Div}_{g-1}(c)/\sim\Big) \cong Pic_{g-1}(c) & \overset{\phi(\cdot)=\cdot\otimes K_c^{-1/2}}{\longrightarrow} & Pic_0(c) \\ \downarrow \mu & & \downarrow\cong \\ W_{g-1} & \overset{\widetilde{\phi}(\cdot)=\cdot-\mu(K_c^{1/2})}{\longrightarrow} & \Theta_c \hookrightarrow Jac(c) \end{array}$$

mapping a degree $g-1$ line bundle $\mathcal{L} = \mathcal{O}_c(D_{g-1}) = K_c^{1/2} \otimes \mathcal{F}$ to the flat bundle $\phi(\mathcal{L}) = \mathcal{F}$ (for all the standard notation cf. appendix B.4). For the family over $Jac(c)$ of $\bar{\partial}$ operators

$$\mathcal{C}^\infty\Big(c, \mathcal{O}_c(D_{g-1})\Big) \xrightarrow{\bar{\partial}} \mathcal{C}^\infty\Big(c, \mathcal{O}_c(D_{g-1}) \otimes \Omega^{0,1}\Big) \tag{2.7}$$

one has, noting that $\theta(\cdot, \Omega_c)$ vanishes just if $\ker \bar{\partial} \; = \; H^0(c, \mathcal{O}_c(D_{g-1}))$ is non-trivial,

$$DET \;\bar{\partial}_c \;\; = \;\; \mathcal{O}_{Jac(c)}\left(\Theta_c\right) \tag{2.8}$$

Note that this concerns the 'vertical' direction z related to the line bundle or divisor and not the 'horizontal' direction related to the parameter Ω_c of the varying curve, cf. above.

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3 The case of base curves in elliptically fibered X and fibrewise semistable bundles

Let $\pi : X \to B$ be a Calabi-Yau space, elliptically fibered over the base B (embedded via a section σ) and V an SU(n) bundle, semistable on the generic elliptic fibre F, so V arises by the spectral cover construction, cf. appendix A.3 for notations. Associated to V is a spectral surface C (an n-fold ramified cover of B) of class $n\sigma + \pi^*\eta$ and let the spectral twist parameter $\lambda \in \frac{1}{2}\mathbf{Z}$ be kept fixed (we assume $h^{1,0}(C) = 0$, as C may be ample, say).

Furthermore we consider the case that the (isolated) rational instanton curve b lies in the base B; its respective normal bundles in B and the elliptic surface $\mathcal{E} = \pi^{-1} b$ over b have first Chern classes given by the respective self-intersection numbers $c_1 (N_B b) = b^2$ and $c_1 (N_{\mathcal{E}} b) = s^2 = -\chi$ with $s = \sigma|_{\mathcal{E}}$ and $\chi := c_1 \cdot b$ (so $\mathcal{T} := K_B^{-1}$ becomes $\mathcal{O}_b(\chi)$ after restriction to b), cf. [9]; by restriction one gets the spectral curve $c \subset \mathcal{E}$ over b of class $ns + rF$ where $r = \eta\, b$ (these cohomological data will be held fixed throughout).

The pfaffian factors through the restriction maps between the bundle moduli spaces

$$\mathcal{M}_X (V) \longrightarrow \mathcal{M}_{\mathcal{E}} (V_{\mathcal{E}}) \quad \longrightarrow \mathcal{M}_b (V_b) \tag{3.1}$$

The difference between bundles $V_{\mathcal{E}}$ (or V_b) *defined from the outset just over* $\mathcal{E}$ *(or* b*)* and the restrictions $V|_{\mathcal{E}}$ (or $V|_b$) of our bundle V (defined over X), i.e. between the full moduli spaces on the right and the image(s) from the left, will be important (cf. section 3.2).

3.1 The family of chiral Dirac operators in the elliptic case

The base space of the Pfaffian line bundle will be the moduli space $\mathcal{M}_X(V)$ of the spectral bundle V (the curve b is held fixed). For bundles defined via the spectral cover construcion the whole situation will be pulled-back to an situation uplifted over b. This happens by consideration of a line bundle (fixed up to a discrete choice $\lambda \in \frac{1}{2}\mathbf{Z}$) over a cover curve c of b; this curve c varies in accordance with a motion in the bundle moduli space $\mathcal{M}_X(V)$. The considerations of section 2 thus apply now just on the uplifted level.

We fix the parameters n, η, λ and consider from now on just the corresponding component $\mathcal{M}_X^{(\lambda)}(V)$. We have a family of bundles (with m the varying modulus in $\mathcal{M}_X^{(\lambda)}(V)$)

$$\begin{array}{l} \quad\underline{\mathcal{S}} \\ \quad\downarrow\ S_b \otimes V_m|_b \\ \mathcal{M}_X^{(\lambda)}(V) \end{array} \tag{3.2}$$

cf. (2.4). The base can be identified with the linear system $|C| = \mathbf{P}H^0(X, \mathcal{O}_X(C))$.

In this preliminary subsection we just want to give a first idea how the general approach is expressed in our concrete case. For spectral bundles one is led to consider the moduli of the spectral curve c over b. Then one relates (3.2) to the family

$$\begin{array}{l} \quad\mathcal{S} \\ \quad\downarrow\ S_c \otimes \mathcal{F}_c \\ \mathcal{M}_{\mathcal{E}}^{(\lambda)}(V|_{\mathcal{E}}) \end{array} \tag{3.3}$$

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The base will be identified with the linear system $|c| = \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(c))$ as here now the curve $c \subset \mathcal{E}$ varies; the flat line bundle $\mathcal{F}$ varies along with the concrete specific curve $c' = (s) \in |c|$, for $s \in H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(c))$, by restricting a universal line bundle on $\mathcal{E}$ to the respective curve (later, cf. remark after (4.8) and footnotes 16 and 22, we will point to some subtleties pertaining to the question where precisely which line bundle is defined).

One has with the n-fold cover curve c of b in the elliptic surface $\mathcal{E}$ and the flat line bundle $\mathcal{F} \in Pic^0(c)$ on c (note the corresponding facts $c_1(V|_b) = 0$ and $c_1(\mathcal{F}) = 0$) that

$$S_-(b) \otimes V|_b = \pi_{c*}\Big(S_-(c) \otimes \mathcal{F}\Big) \tag{3.4}$$

This relation has as a consequence the identification

$$\Gamma\Big(b, S_-(b) \otimes V_b\Big) \cong \Gamma\Big(c, S_-(c) \otimes \mathcal{F}\Big) \tag{3.5}$$

(the question of spin structures will be discussed later). As contribution of b to the superpotential just means absence of non-trivial holomorphic sections on the left hand side, one finds a precise condition (cf. below) in the moduli of c: the continuous moduli of V stem from external motions of the spectral surface C in X which in $\mathcal{E}$ map to $\mathcal{M}_\mathcal{E}(c) = \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_\mathcal{E}(c)) = |c|$; these external motions of c in $\mathcal{E}$ are described by the coefficients of its defining polynomial, and they map also to the intrinsic moduli of c

$$\mathbf{P}H^0\Big(X, \mathcal{O}_X(C)\Big) \longrightarrow \mathbf{P}H^0\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}(c)\Big) \longrightarrow \mathcal{M}_g \tag{3.6}$$

The fixed line bundle $\widetilde{\mathcal{L}} = l(-F)$ (for a fixed chosen λ) associates to the varying $c' \in |c|$ an element $\mathcal{L}_{c'} = l(-F)|_{c'} \in Pic^0(c')$. Consideration of the sections in (3.5) therefore leads, using the map $\mu : \ Pic^0(c) \ \longrightarrow \ Jac(c)$, to the criterion for contribution to the superpotential (both sides taken at a specific curve $c' \in \mathcal{M}_\mathcal{E}(c)$; for details cf. section 6)

$$W_b \neq 0 \Longleftrightarrow \theta\Big(\mu(\mathcal{F})\Big) \neq 0 \tag{3.7}$$

3.2 Structure of the moduli space of the bundles over X

Let us recall the structure of the moduli space of spectral bundles. Just as in eight dimensions the spectral points on the elliptic curve represent the degrees of freedom of the bundle, one expects by adiabatic extension that moduli arise from the deformations of the spectral object in its ambient space; the number of the deformations in $\mathbf{P}H^0(X, \mathcal{O}_X(C))$ is then $h^{d,0}(C)$ (cf. below) for a d-dimensional spectral object in an ambient *Calabi-Yau space* as its normal bundle equals then its canonical bundle. In studies of $N = 2$ string-duality in six dimensions that ambient space was a $K3$ surface, elliptically fibered over a $\mathbf{P^1}$; the genus of the spectral curve equals then the quaternionic dimension of the full quaternionic bundle moduli space, given by the total space of the fibration below in (4.2).

Consider the corresponding structure of the moduli space for the bundle V defined over the whole of X. The moduli space of V shows a fibration structure (for $\mathcal{A}_B$ cf. below)

$$\begin{array}{l} \mathcal{M}_X(V) \\ \quad\downarrow \ H^1(B, \mathcal{A}_B) \\ \mathbf{P}H^0\Big(X, \mathcal{O}_X(C)\Big) \end{array} \tag{3.8}$$

We consider first only the tangent space to this fibration, i.e. first-order deformations.[7]

Beginning with the base in (3.8) note that after tensoring the short exact sequence $0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$ with $\mathcal{O}_X(C)$ and taking cohomology one gets the isomorphism[8] $T\mathbf{P}H^0\Big(X, \mathcal{O}_X(C)\Big) \cong H^0(C, N_C) \cong H^0(C, K_C)$, cf. above. On this tangential level the fibration (3.8) arises from the Leray spectral sequence

$$\begin{array}{ll} H^1(X, ad\, V) & \\ \quad\downarrow\; H^1(B, \pi_*\, ad\, V) & \cong H^1(C, \mathcal{O}_C) \\ H^0(B, R^1\, \pi_*\, ad\, V) & \cong H^0(C, K_C) \end{array} \tag{3.9}$$

(using the identification of the abstract space $\pi_*\, ad\, V$ with $\pi_{C*}\, \mathcal{O}_C$ and, concerning the space of deformations of C in X, the identification of $R^1\, \pi_*\, ad\, V$ with[9] $\pi_*\, \mathcal{O}_X(C) \cong \oplus_{\substack{i=0\\ i\neq 1}}^n \mathcal{M} \otimes \mathcal{T}^{-i}$ which in turn is identified with $\pi_{C*}\, \mathcal{O}_X(C)|_C \cong \pi_{C*}\, K_C$).

Let us go beyond the consideration local in the fibre made here and describe how the other components of the full fibre $\{\gamma\} \cong \mathbf{Z}$ (parametrised by λ) enter the picture, cf. [8].

So let us discuss the twisting data which occur when piecing together along B bundles over the fibers. Let $\mathcal{M}_F$ be the moduli space of semistable SU(n)-bundles over F, $\mathcal{M}_{X/B}$ the relative object (fibered over B) and Ξ the universal bundle over[10] $F \times \mathcal{M}_F^0$ or $X \times_B \mathcal{M}_{X/B}^0$. The bundle V over X, fiberwise[11] semistable, gives a section $\tilde{s}$ of $\mathcal{M}_{X/B}^0 \rightarrow B$. Conversely one tries to build V from $\tilde{s}$ by pulling back a universal bundle.

On the fibre there is, associated to Ξ, an abelian group scheme of automorphism groups $\underline{Aut}(\Xi)$ over $\mathcal{M}_F^0$ (of associated sheaf of sections $\mathcal{A}$). The set of universal bundles over $F \times \mathcal{M}_F^0$ is[12] rotated through under $H^1(\mathcal{M}_F^0, \mathcal{A})$, or $H^1(B, \mathcal{A}_B(\tilde{s}))$ in the relative version.

The twisting data $H^1(B, \mathcal{A}_B)$ are fibered over its discrete part $\gamma\mathbf{Z}$ by the continuous relative jacobian $Jac(C/B)$, cf. [2]. This is itself the relative version of the fibration of the Picard group (the moduli space of line bundles) over its discrete part (characterised by the first Chern class) by the set of flat bundles (parametrized by a complex torus)

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^1(B, \mathcal{A}_B) & \rightarrow & Pic\, C & \rightarrow & Pic\, B & \rightarrow & 0 \\ & & \downarrow\; Jac(C/B) & & \downarrow\; Pic_0\, C & & \downarrow\; Pic_0\, B & & \\ 0 & \rightarrow & \{\gamma\} \cong \mathbf{Z} & \rightarrow & H^{1,1}(C, \mathbf{Z}) & \rightarrow & H^{1,1}(B, \mathbf{Z}) & \rightarrow & 0 \end{array}$$

The B considered here have trivial $Pic_0(B)$. As $\pi_{C\,*}(c_1(L))$ is fixed by the condition $c_1(V) = 0$ the possible L's are parametrized[13] by a discrete part encoded in λ and a continuous part from $Jac(C/B) \cong Pic_0(C) \cong Jac(C)$. In case C is ample (positive) one

[7] Which are unobstructed [3], however, for C smooth and B rational.

[8] As $h^{0,1}(X) = 0$; when applying the same argument to $\mathcal{E}$ and c therefore $h^{0,1}(\mathcal{E}) = 0$ will be relevant.

[9] Cf. for (3.9) at this stage the tangential map of (A.6) and the dimension computation (A.12) from coefficients of the defining equation of C.

[10] The superscript 0 denotes the smooth locus where a universal object exists locally.

[11] Over the open subset of B over which lie smooth fibers; the mentioned section is also over that subset.

[12] If the obstruction in $H^2(\mathcal{M}_F^0, \mathcal{A})$ vanishes.

[13] We restrict us to the case that $Pic(C)$ is generated by the restrictions of elements of $Pic(X)$.

has $\pi_1(C) \cong 1$ such that $h^{1,0}(C) = 0$, *as we assume throughout*. So over X no continuous moduli appear as $Pic_0(C) \cong Jac(C)$ will not enter the story, only the discrete choice in $\ker \pi_{C*} : H^{1,1}(C) \to H^{1,1}(B)$ remains. For simplificity we assume the generic situation[13] that the only classes in this kernel are those visible already on X, i.e. arise by restriction; then the remaining discrete part is parametrized by a number $\lambda \in \frac{1}{2}\mathbf{Z}$ (cf. appendix A.3).

So, given our assumptions, to let vary a bundle V over its moduli space $\mathcal{M}_X(V)$ comes down (once the discrete parameter $\lambda \in \frac{1}{2}\mathbf{Z}$ is fixed) to let vary the concrete surface given by the zero set $(\underline{s})$ in the linear system $|C| = \mathbf{P}H^0(X, \mathcal{O}_X(C))$ (for $\underline{s} \in H^0(X, \mathcal{O}_X(C))$). Let denote $\underline{S}_\lambda$ the corresponding section of the fibration (3.8) (whose fibre is just $\{\gamma\} \cong \mathbf{Z}$)

$$\underline{S}_\lambda \;\in\; \Gamma\Big(\mathbf{P}H^0\big(X, \mathcal{O}_X(C)\big), \mathcal{M}_X(V)\Big) \tag{3.10}$$

4 The moduli space of the restricted bundles

Recall that we consider the following restriction map between the moduli spaces

$$\mathcal{M}_X(V) \longrightarrow \mathcal{M}_\mathcal{E}(V_\mathcal{E}) \tag{4.1}$$

Before considering the *image of this map* (the part of the right hand side relevant for us) in more detail, we note that the full moduli space $\mathcal{M}_\mathcal{E}(V_\mathcal{E})$ of spectral $\mathrm{SU}(n)$ bundles *which are defined from the outset just over* $\mathcal{E}$ possesses the fibration (with $Jac(c_t) \cong Pic_0(c_t)$)

$$\begin{array}{c} \mathcal{M}_\mathcal{E}(V_\mathcal{E}) \\ \downarrow \; Jac(\cdot) \\ \mathbf{P}H^0\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}(c)\Big) \end{array} \tag{4.2}$$

(in the base a section $s \neq 0$ of $\mathcal{O}_\mathcal{E}(c)$ defines a curve given by its zero set (s)).

Note that, unlike the situation for a two-dimensional base B considered above, here $\mathcal{E}$ will not be a Calabi-Yau ambient space for the spectral object c (as $\mathcal{E}$ will not be a $K3$ surface in general). The former $\mathcal{T} = K_B^{-1}$, the line bundle relevant to the description of the elliptic fibration, becomes inside $\mathcal{E}$ now $\mathcal{T}|_b = \mathcal{O}_b(\chi) = \mathcal{O}_b(1)$ as $\chi = 1$ for b isolated.

By the restriction map $V \longrightarrow V|_\mathcal{E}$ the following phenomenon occurs: the moduli space of bundles over elliptically fibered spaces shows a significant difference between the cases where the base is either B or b. Whereas in the spectral *surface* case the continuous part of the twist data (the fibre of the moduli space) vanishes under generic conditions (like C being ample) this is not the case for a spectral *curve* which comes always naturally equipped with its Jacobian (but here no discrete twist arises). This has the following consequence: a bundle V over X, considered to vary over its moduli space, is described just by a *constant* section $\underline{S}_\lambda$ of its fibration (3.8). However, its restriction to $\mathcal{E}$

$$\begin{array}{ccc} \mathcal{M}_X(V_X) & \longrightarrow & \mathcal{M}_\mathcal{E}(V_\mathcal{E}) \\ \downarrow \; \{\gamma\} \cong \mathbf{Z} & & \downarrow \; Jac(\cdot) \\ \mathbf{P}H^0\Big(X, \mathcal{O}_X(C)\Big) & \longrightarrow & \mathbf{P}H^0\Big(\mathcal{E}, \mathcal{O}_\mathcal{E}(c)\Big) \end{array} \tag{4.3}$$

is given by a section S_λ of a moduli space fibration (4.2) with the fibre no longer discrete

$$\Gamma\Big(\mathbf{P}H^0\big(X,\mathcal{O}_X(C)\big),\mathcal{M}_X(V)\Big)\ni\underline{S}_\lambda\longrightarrow S_\lambda\in\Gamma\Big(\mathbf{P}H^0\big(\mathcal{E},\mathcal{O}_\mathcal{E}(c)\big),\mathcal{M}_\mathcal{E}(V_\mathcal{E})\Big)\tag{4.4}$$

So, for a bundle *defined a priori just over* $\mathcal{E}$ the total space $\mathcal{M}_\mathcal{E}(V_\mathcal{E})$ in the fibration (4.2) is relevant as describing a varying object; for our case, with a $V_\mathcal{E}$ *arising as restriction* $V_X|_\mathcal{E}$ *where* $V_{(X)}$ *is defined already over* X, the image of the restriction map (4.1) rather is (holding fixed $\lambda\in\frac{1}{2}\mathbf{Z}$, giving a trivial fibre in (3.8)) just one section S_λ of (4.2).

4.1 Decomposition of $\mathcal{L}$ relative to a spin choice

The moduli of V, considered over X, consist, besides the fixed discrete choice $\lambda\in\frac{1}{2}\mathbf{Z}$, in the motions of C in X, i.e. they are given (up to an overall rescaling) by the coefficients of the sections a_i in the defining polynomial, cf. appendix A.1; the world-sheet instanton superpotential W_b will depend only on their restrictions to b. For B the Hirzebruch surface $\mathbf{F_k}$, say, the a_i are certain polynomials $a_i(z_1,z_2)$ (whose coefficients are essentially the moduli) and for W_b only the coefficients are concerned which remain after setting $z_2=0$ (i.e. after restricting from B to b); in other words, W_b depends effectively only on the image of a section to the restriction map of $\mathcal{M}_X(V)$ on its image in $\mathcal{M}_\mathcal{E}(V_\mathcal{E})$.

Before proceeding further let us keep on record here two important line bundles

$$\Lambda=\mathcal{O}_\mathcal{E}\Big(ns-(r-n)F\Big)\tag{4.5}$$

$$K_c=\mathcal{O}_\mathcal{E}\Big(ns+(r-1)F\Big)\Big|_c\tag{4.6}$$

The importance of Λ stems from the fact that[14] it generates, among the bundles $\mathcal{O}_\mathcal{E}(ps+qF)$ on $\mathcal{E}$ (i.e. the ones which come from X), those which become flat on $c\simeq ns+rF$.

For a spectral cover bundle $V=p_*(p_C^*L\otimes\mathcal{P})$ over X, with $L=\underline{L}|_C$ a line bundle on C arising as restriction from X, one has for $\widetilde{\mathcal{L}}=l(-F)$ with $l=\underline{L}|_\mathcal{E}$

$$\widetilde{\mathcal{L}}=\begin{cases}\mathcal{O}_\mathcal{E}\Big(\frac{1}{2}\Big[ns+(r-1)F\Big]\Big)\otimes\mathcal{O}_\mathcal{E}\Big(\lambda\Big[ns-(r-n)F\Big]\Big) & \text{for } n \text{ even } (\rightarrow\lambda\in\mathbf{Z}, r \text{ odd})\\ \mathcal{O}_\mathcal{E}\Big((\lambda+\frac{1}{2})ns+\Big[(\lambda n-\frac{1}{2})-(\lambda-\frac{1}{2})r\Big]F\Big) & \text{for } n \text{ odd } (\rightarrow\lambda\in\frac{1}{2}+\mathbf{Z})\end{cases}\tag{4.7}$$

(the implications for n even/odd follow by the remark after (A.24); so the expressions for $\widetilde{\mathcal{L}}$ exist). For us $V_\mathcal{E}$ is $V_{(X)}|_\mathcal{E}$ with $\widetilde{\mathcal{L}}|_c=\mathcal{L}$ a flat twist of a theta characteristic

$$\mathcal{L}=K_c^{1/2}\otimes\mathcal{F}\tag{4.8}$$

Here for n even, cf. (4.7), we use the standard representative $\mathcal{O}_\mathcal{E}(\frac{1}{2}[ns+(r-1)F])|_c$ for $K_c^{1/2}$ coming already from $\mathcal{E}$ (canonical "global spin choice", here 'global' refers to the variation of c in $\mathcal{M}_\mathcal{E}(c)$); as $\lambda\in\mathbf{Z}$ also $\mathcal{F}=\Lambda|_c^\lambda=\mathcal{O}_c(G|_c)$ exists individually. For n odd, considered *on* $\mathcal{E}$, only the combination in (4.7) exists; however, c being a curve, a square root of the canonical bundle will always exist, though having a $Pic_0^2(c)$ ambiguity; we will consider a globalization over $\mathcal{M}_\mathcal{E}(c)$ of such a choice as part of the data (a non-unique "global spin

[14] For $gcd(n,r-n)=1$, which is fulfilled automatically in the cases of application (where $n\leq 5$).

choice"), keeping in mind its indicated non-uniqueness. A flat bundle $\mathcal{F}_c$ will then also exist (as $\mathcal{L}$ exists independent of choices), depending on the choice indicated. Then, once a global spin choice is made, in $\mathcal{F}_{c_t}$ it is the variation inside $\mathcal{E}$ of the respective concrete curve c_t in the linear system $|c| = \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)) = \mathcal{M}_{\mathcal{E}}(c)$, which gives the respective position of $\mathcal{F}_{c_t}$ in a Jacobian $Jac(c_t)$.

4.2 Dimension of the moduli space and the divisor $(Pfaff)$

For c positive one finds by the index theorem (the second relation holds in general)

$$h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)) = n(r+1) - \left(\frac{n(n+1)}{2} - 1\right)\chi \tag{4.9}$$

$$\deg\, l(-F)|_c = \deg K_c^{1/2} = n(r-1) - \left(\frac{n(n-1)}{2}\right)\chi \tag{4.10}$$

The moduli space of external motions of c has dimension $h^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big) - 1$ and for b isolated we have $\chi = 1$ [9] such that one gets then (where $\dim Jac(c) = \deg\ K_c^{1/2} + 1$)

$$\dim \mathcal{M}_{\mathcal{E}}(c) = n\left(r - \frac{n-1}{2}\right) = g - 1 + n \tag{4.11}$$

$$\dim Jac(c) = n\left(r - \frac{n+1}{2}\right) + 1 = g \tag{4.12}$$

$$\dim \mathcal{M}_{g_c} = 3n\left(r - \frac{n+1}{2}\right) = 3g - 3 \tag{4.13}$$

A concrete basis ω_α, $\alpha = 1, \ldots, g$, of $H^{1,0}(c_t)$ arises by taking Poincare residues of elements in $\Omega^2_{\mathcal{E}}(c)$, written locally as $\omega = \frac{h}{w} du \wedge dX$ where $u = u_1/u_2$ and $X = x/z$ are affine coordinates on b and F, resp. (on $\mathcal{E} - s - F$, fibered by $F - \{p_0\}$ over $b - \{q_0\}$) and $w = 0$ the spectral curve equation. Because $K_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}(-F)$ one gets $h = h_0 + h_2 x + h_3 y + \cdots + h_n x^{n/2} \in H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(ns + (r-1)F))$ globally[15] such that $\dim \mathcal{M}_{\mathcal{E}}(c) = d(n,r) - 1$ and $g = d(n, r-1)$ where $d(n,r) := h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$; concretely one gets $\omega_\alpha = \frac{h_\alpha}{\partial_u w} dX = -\frac{h_\alpha}{\partial_X w} du$.

The fact that $\dim \mathcal{M}_{\mathcal{E}}(c) \neq \dim Jac(c)$ reflects the fact that $\mathcal{E}$ is not a $K3$ surface, so no hyperkähler structure for $\mathcal{M}_{\mathcal{E}}(V_{\mathcal{E}})$ arises (the equality of dimensions is recovered, however, for the Calabi-Yau space over $\mathbf{F_0}$ where $\mathcal{E}$ becomes $K3$ and $\chi = 2$ gives $n(r-n) + 1$).

The fact that $\dim \mathcal{M}_{\mathcal{E}}(c) < \dim \mathcal{M}_g$ reflects the fact that the curves c of class $ns + rF$ and of genus g_c arising in $\mathcal{E}$ are not the most general curves of genus g (cf. the non-genericity of plane curves in $\mathbf{P^2}$). For example, because of the n-fold covering of b, they carry a certain r-dimensional system g^r_d of divisors of degree d, here a g^1_n; however the generic genus g curve has such a system only if $g \leq 2n - 2$, what contradicts the genus (4.12) (we always have $r > n$, cf. appendix A.3). The space $\mathcal{M}^1_{g,n}$ of curves having a g^1_n has dimension $2g + 2n - 5$ if, as in our case, $n \leq \frac{g+2}{2}$ (note, for comparison, $\dim \mathcal{M}^{\text{hyp}}_g = 2g - 1$).

The degree of the Pfaffian polynomial $Pfaff = \det\ \iota_1$ is given by $h^1(\mathcal{E}, l(-F)) = (\lambda^2 - \frac{1}{4})n(r - \frac{n}{2}) - 1$. Therefore the $W_b = 0$ locus is given by a divisor $(Pfaff) \subset \mathbf{P}H^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big)$ of degree $2k - n - 1$ (for $\lambda = 3/2$, say) inside a $\mathbf{P}^k$ where $k = n(r - \frac{n-1}{2})$.

[15] For n even, say, cf. appendix 4.4; the coefficient functions $h_i(u_1, u_2)$ have degree $r - 1 - i$; the h_α with $\alpha = 1, \ldots, g = d(n, r-1)$ are suitably enumerated also by $h_\alpha = h_{ij}$ with $i = 0, 2, 3, \ldots, n, j = 0, 1, \ldots, r - 1 - i$.

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4.3 Special loci in the moduli space: $\Sigma \subset (Pfaff)$

Recall that $\mathcal{L} = K_c^{1/2} \otimes \Lambda|_c^\lambda$ (cf. (4.7), remarks there and footn. 16) where $\lambda \in \frac{1}{2}\mathbf{Z}$ and

$$\Lambda = \mathcal{O}_\mathcal{E}(ns - (r-n)F) \qquad \text{with} \quad \Lambda|_{c_t} \in Pic_0(c_t) \tag{4.14}$$

Define the subset Σ_k of $\mathcal{M}_\mathcal{E}(c)$ where $\Lambda|_{c_t}$ becomes a k-torsion element. So (cf. appendix B.3)

$$\Sigma_{2\lambda} = \left\{ t \in \mathcal{M}_\mathcal{E}(c) \,\middle|\, \Lambda|_{c_t} \in Pic_0^{2\lambda}(c_t) \ \ (\Longleftrightarrow \ \mathcal{L}_{c_t} \in \mathcal{S}p(c_t)) \right\} \tag{4.15}$$

$$\Sigma = \left\{ t \in \mathcal{M}_\mathcal{E}(c) \,\middle|\, \Lambda|_{c_t} \cong \mathcal{O}_{c_t} \right\} \subset \Sigma_{2\lambda} \tag{4.16}$$

(for the definition of $\Sigma = \Sigma_1$ cf. section 3.2.1 of [9]). $\Sigma_{2\lambda}$ is where $\mathcal{L}_{c_t}^2 \cong K_{c_t}$ holds; this is just the locus where $\tau^* V|_\mathcal{E} \cong V^*|_\mathcal{E}$ [1] (for τ cf. appendix A.2), such that, by $(\tau^* V)|_B = V|_B$,

$$t \in \Sigma_{2\lambda} \Longrightarrow \quad V|_b \cong V^*|_b \tag{4.17}$$

$$t \in \Sigma \quad \Longrightarrow Pfaff(t) = 0 \tag{4.18}$$

Here (4.18) is obvious as the effectivity of D in $\mathcal{L} = \mathcal{O}_c(D)$ is sufficient for $Pfaff$ to vanish (cf. also (6.2)): now one gets[16] along Σ that (for the parities cf. (A.24))

$$\mathcal{L} \overset{\Sigma}{\to} \begin{cases} K_{c_t}^{1/2} = \mathcal{O}_\mathcal{E}\left(\frac{n}{2}s + \frac{r-1}{2}F\right)\Big|_{c_t} & \text{for } \lambda \in \mathbf{Z} \ (\to n \,\text{even}, r \,\text{odd}) \\ \mathcal{O}_\mathcal{E}\left(\left[(\lambda+\frac{1}{2})(r-n)+\beta\right]F\right)\Big|_{c_t} = \mathcal{O}_\mathcal{E}\left(\left[r - \frac{n+1}{2}\right]F\right)\Big|_{c_t} & \text{for } \lambda \in \frac{1}{2} + \mathbf{Z} \ (\to n \,\text{odd}) \end{cases} \tag{4.19}$$

So we have the array of inclusions (we assume we are not in a case where $Pfaff \equiv 0$)

$$\begin{tikzcd} & & (Pfaff) \arrow[dr] & \\ \Sigma \arrow[r] & \Sigma_{2\lambda} \cap (Pfaff) \arrow[ur] \arrow[dr] & & \mathcal{M}_\mathcal{E}(c) \\ & & \Sigma_{2\lambda} \arrow[ur] & \end{tikzcd} \tag{4.20}$$

where $(Pfaff) \hookrightarrow \mathcal{M}_\mathcal{E}(c)$ has codimension one (for the restriction to $\Sigma_{2\lambda}$ cf. Rem. (2)).

[16] Actually one gets the same conclusion for $\lambda \in \frac{1}{2} + \mathbf{Z}$ as in the case of $\lambda \in \mathbf{Z}$, despite first appearence: note that not only has $\frac{1}{2} K_{c_t}$ the *integral* degree $n(r - \frac{n+1}{2})$ - clearly a necessary condition for the existence of the bundle $\mathcal{O}_c(\frac{1}{2}(ns|_c + (r-1)F|_c))$ - but, c_t being a curve, the bundle $K_{c_t}^{1/2}$ will always exist; therefore $\Lambda|_{c_t}^\lambda$ will exist as well because the combination $\mathcal{L}_{c_t} = K_{c_t}^{1/2} \otimes \Lambda|_{c_t}^\lambda$ exists (this because $\mathcal{L}_{c_t}$ is the restriction to c_t of the line bundle $\widetilde{\mathcal{L}}$ which exists already on $\mathcal{E}$, by our standing integrality assumptions, cf. appendix A.3); now, however, we do not know that $\Lambda|_{c_t}^\lambda \cong \mathcal{O}_{c_t}$ but rather only that $\Lambda|_{c_t}^\lambda \in Pic_0^2(c_t)$ (actually this expression is not even defined more precisely because similarly we do not know precisely which square root of K_{c_t} here the $K_{c_t}^{1/2}$ is, as it will not arise by restriction from $\mathcal{E}$ as was the case for λ integral; that is, here the individual factors would depend essentially on a spin choice made); so $\mathcal{L}$ becomes now (instead of the explicit $K_{c_t}^{1/2}$ in (4.19) for $\lambda \in \mathbf{Z}$) for $\lambda \in \frac{1}{2} + \mathbf{Z}$ a different theta characteristic (spin bundle structure, i.e. square root of K_{c_t}, cf. appendix B.3) $K_{c_t}^{1/2} \otimes \Lambda|_{c_t}^\lambda$ which means here concretely the second line of (4.7) restricted to c; then the argument can be completed as indicated in (4.19), with the ultimate result being independent of any potential spin choices.

Remarks. Let us add some remarks on a type of set like $\Sigma_{2\lambda}$.

(1) $\Sigma_{2\lambda} \subset \mathcal{M}_{\mathcal{E}}(c)$ is the locus where $\Lambda|_{c_t}^{\lambda} \in Pic_0^2(c_t)$; here $\Lambda|_{c_t}^{\lambda}$ is for n odd only *defined* up to $Pic_0^2(c_t)$ as then also the factor $K_c^{1/2}$, to be split off from $\mathcal{L}_{c_t}$, is defined only up to that ambiguity (here $\lambda \in \frac{1}{2}\mathbf{Z}$ whereas above we had $k \in \mathbf{Z}$; cf. footn. 16).

(2) Note that $\Sigma_{2\lambda}$ decomposes as $\Sigma_{2\lambda}^{+} \dot\cup \Sigma_{2\lambda}^{-}$ according to the parity of $\mathcal{L}_{c_t}$ (that is of $h^0(c_t, \mathcal{L}_{c_t})$) such that one gets

$$\Sigma_{2\lambda} \cap (Pfaff) = \left(\Sigma_{2\lambda}^{+} \cap (Pfaff)\right) \;\dot\cup\; \Sigma_{2\lambda}^{-} \tag{4.21}$$

This might be compared to (B.16); accordingly one has under the period map in (6.6)

$$\Sigma_{2\lambda}^{+} \cap (Pfaff) \xrightarrow{\Pi} \mathcal{M}_g^1 \subset \mathcal{M}_g \tag{4.22}$$

So it is the component $\Sigma_{2\lambda}^{+} \cap (Pfaff)$ which has codimension one in $\Sigma_{2\lambda}^{+}$ (or $\Sigma_{2\lambda}$).

(3) The obvious relation $\Sigma \subset \Sigma_{2\lambda}$ is a special case of the general fact $\Sigma_{k'} \subset \Sigma_k$ for $k'|k$. Note, however, that here the codimension of $\Sigma_{k'}$ in Σ_k cannot be > 0: in that case one would have a continuous deformation from elements of $\Sigma_k - \Sigma_{k'}$ to elements of $\Sigma_{k'}$ which is impossible (as elements of strictly higher torsion order cannot be deformed, inside Σ_k, to elements of lower torsion order); so Σ_k is rather reducible with $\Sigma_{k'}$ a component. So Σ is a component of $\Sigma_{2\lambda}$; note also that $\Sigma_k \cap \Sigma_{k'} = \Sigma_{gcd(k,k')}$; $\Sigma_{2\lambda}$ has components

$$\Sigma_{2\lambda} = \dot{\bigcup}_{k'|2\lambda} \Sigma_{k'}^{\text{strict}} \tag{4.23}$$

comprising elements of $\Sigma_{k'}$ of *strict* torsion order k'; for p prime, say, one has $\Sigma_p = \Sigma \,\dot\cup\, \Sigma_p^{\text{strict}}$ where $\Sigma \subset (Pfaff)$; compare the relation $\Sigma_p = \Sigma_p^{+} \dot\cup \Sigma_p^{-}$ with $\Sigma_p^{-} \subset (Pfaff)$.

(4) Besides Σ one has, via the same argument, further components lying in $(Pfaff)$

$$\Sigma_{\lambda} \subset (Pfaff) \qquad \text{for } n \equiv 0\,(2) \tag{4.24}$$

$$\Sigma_{\lambda+\frac{1}{2}} \subset (Pfaff) \qquad \text{for } n \not\equiv 0\,(2) \tag{4.25}$$

This follows as the specialisations in (4.19) still hold in these larger loci. On a further locus $\mathcal{L}$ becomes $\mathcal{O}_{\mathcal{E}}(ns + \frac{n-1}{2}F)|_{c_t}$, again the line bundle of an effective divisor,

$$\Sigma_{\lambda-\frac{1}{2}} \subset (Pfaff) \qquad \text{for } n \not\equiv 0\,(2) \tag{4.26}$$

There are also special cases like $\Sigma_{\lambda-\frac{3}{2}} \subset (Pfaff)$ for $n = 3, r = 4$ and $n = 5, r = 6, 7$. A further special case is $\lambda = 9/2$ with $\Sigma_2 \subset (Pfaff)$ for $n = 5, r = 6$.

4.4 A further explicit locus in the moduli space: $\mathcal{R} \subset \Sigma$

We will denote points of $\mathcal{E} \subset \mathcal{W}_b$ by pairs $\{p, u\}$ where p and u are coordinates of $\mathbf{P^2}$ and $b \cong \mathbf{P^1}$, respectively. One gets the divisors (with $p_0 = (0, 1, 0)$ and $a_n(u_j) = 0$)

$$s|_c = \sum_{j=1}^{r-n} \{p_0, u_j\} \tag{4.27}$$

$$F_u|_c = \sum_{i=1}^{n} \{q_i, u\} \tag{4.28}$$

Let $a_n = \prod_{j=1}^{r-n} a_n^{(j)}$ be a decomposition in linear factors with $a_n^{(j)}(u_j) = 0$ and let

$$R_i^{(j)} := Res(a_i, a_n^{(j)}) \tag{4.29}$$

denote the resultant (cf. appendix I.1 of [9]) for $i = 2, \ldots, n-1$ and $n > 2$. If (cf. section 3.2.2 of [9]) $R_i^{(j)} = 0$, i.e. $a_n^{(j)}|a_i$, for all $i = 2, \ldots, n-1$, *for one specific* j then

$$ns|_c = \sum_{k=1}^{r-n} \{np_0, u_k\} \;=\; F_{u_j}|_c + \sum_{\substack{k=1\\k\neq j}}^{r-n} \{np_0, u_k\} \tag{4.30}$$

as then $q \in F_{u_j}|_c$ is forced to have $z = 0$, i.e. to be p_0 (here p_0 is already among the n points $\{q_i^{(j)}, u_j\}$, say for $i = 1$; then the points for $i = 2, \ldots, n-1$ are forced to be at p_0 and the final one follows to be there as well because the points sum up to zero, represented by p_0, in the group law). Thus $(ns - F_{u_j})|_c$ = effective and so $(ns - F)|_c \sim$ effective. Posing $n-2$ conditions leads to a locus of codimension $n-2$. If, however, $R_i^{(j)} = 0$ for all $i = 2, \ldots, n-1$ and even *for all* $j = 1, \ldots, r-n$ then one has (cf. (A.16))

$$ns|_c = \sum_{j=1}^{r-n} F_{u_j} \sim (r-n)F|_c \qquad \Longrightarrow \qquad G|_c \sim 0 \tag{4.31}$$

or equivalently $\Lambda|_c = \mathcal{O}_{\mathcal{E}}(G)|_c \cong \mathcal{O}_c$. As posing $(n-2)(r-n)$ conditions is sufficient for $G|_c \sim 0$ the latter will hold at a locus of codimension $\leq (n-2)(r-n)$, cf. also (4.34). So one finds (where as usual $(R_i^{(j)})$ denotes the (vanishing) divisor of the polynomial $R_i^{(j)}$)

$$t \in \Sigma \iff \big(ns - (r-n)F\big)\big|_{c_t} \sim \text{effective} \iff \sum_{j=1}^{r-n}\sum_{i=1}^{n} \{p_0 - q_i^{(j)}, u_j\}_{c_t} \sim \text{effective} \tag{4.32}$$

$$t \in \mathcal{R} := \bigcap_{\substack{i=2,\ldots,n-1\\ j=1,\ldots r-n}} (R_i^{(j)}) \Longrightarrow \sum_{j=1}^{r-n}\sum_{i=1}^{n} \{p_0 - q_i^{(j)}, u_j\}_{c_t} = \text{effective} \tag{4.33}$$

where the effective divisor in the second line is of course the zero divisor (i.e. $F_{u_j}|_c = \{np_0, u_j\}$ for all j). So the locus $\mathcal{R} \subset \mathcal{M}_{\mathcal{E}}(c)$, defined explicitly by the equations $R_i^{(j)} = 0$, is a sublocus of the structurally defined locus Σ, cf. (4.34) below; cf. also appendix A.4.

n	g	$\dim\Sigma \geq \dim\mathcal{R} =$	$\dim\mathcal{M}_{\mathcal{E}}(c)$	$\dim\mathcal{M}^1_{g,n}$	$\dim\mathcal{M}_g^{\mathrm{hyp}}$	$\dim\mathcal{M}_g$
2	$2r-2$	$2r-1$	$2r-1$	$4r-5$	$4r-5$	$6r-9$
3	$3r-5$	$2r$	$3r-3$	$6r-9$	$6r-11$	$9r-18$
4	$4r-9$	$2r+2$	$4r-6$	$8r-15$	$8r-19$	$12r-30$
5	$5r-14$	$2r+5$	$5r-10$	$10r-23$	$10r-29$	$15r-45$

Table 1. Dimensions of some loci in the moduli space.

Above we did some general considerations in $Pic(c) \cong Div(c)/\sim$ leading to the relation $\mathcal{R} \subset \Sigma$. On the other hand (4.18) gave some understanding of non-topological (codim > 0) vanishing loci of $Pfaff$: Σ is contained in the codim 1 locus given by the vanishing divisor $(Pfaff)$ (in [9] we looked for whole components of this reducible divisor). Although the *structurally* defined Σ is not easily described *explicitly* in $\mathcal{M}_{\mathcal{E}}(c)$ we now have identified here a concretely described sublocus: the locus $\mathcal{R}$

$$\mathcal{R} \;\subset\; \Sigma \;\subset\; (Pfaff) \tag{4.34}$$

which is of codimension $(n-2)(r-n)$ in $\mathcal{M}_{\mathcal{E}}(c)$. Then $\dim\Sigma \geq \dim\mathcal{R} = n\frac{n-3}{2} + 2r$ from (4.34) gives in the first few cases the list of table 1 (where $r > n$, cf. appendix A.3).

Above we considered the relation of the structurally defined locus Σ and the explicitly described locus $\mathcal{R}$ in general. Let us now specialise to some important low ranks: first to the somewhat special case of SU(2) bundles and then, below in section 4.6, to our main example class, the SU(3) bundles.

SU(2) bundles $\boldsymbol{\Sigma = \mathcal{M}_{\mathcal{E}}(c)}$, i.e. $\boldsymbol{Pfaff \equiv 0}$. Here one has $a_2(u_j) = 0 \Longrightarrow z = 0$, cf. (A.7), such that $F_{u_j}|_c = \{2p_0, u_j\}$ and so $\Sigma = \mathcal{M}_{\mathcal{E}}(c)$

$$n = 2 \Longrightarrow G|_c \;\sim\; 0, \;\; i.e. \;\Lambda|_c \cong \mathcal{O}_c \tag{4.35}$$

Here, for SU(2) bundles, c is hyperelliptic (note also that $\mathcal{M}_g^{\mathrm{hyp}} = \mathcal{M}^1_{g,2}$) and one has $Pfaff \equiv 0$ as $\Sigma = \mathcal{M}_{\mathcal{E}}(c)$, cf. (4.35)

$$\Sigma = \mathcal{M}_{\mathcal{E}}(c) \Longrightarrow Pfaff \equiv 0 \tag{4.36}$$

Before we go on in section (4.6) to the more specialised considerations in the case of SU(3) bundles, which constitute our main class of examples, we insert in the next subsection some discussion on the question of multiplicity of a factor in the Pfaffian.

4.5 Some remarks about multiplicities

We are interested in the multiplicity of components of the reducible divisor $(Pfaff)$, cf. (4.62), (4.63) below. In refinement of the criterion $Pfaff(t) = 0 \Longleftrightarrow h^0(c_t, \mathcal{L}_{c_t}) \neq 0$, one gets for the order with which $Pfaff = \det\iota_1$ (cf. equ. (1.1)) vanishes at $t = t^*$

$$\mathrm{ord}_{t^*} \det\iota_1 \;\geq\; \dim\ker\iota_1|_{t=t^*} = h^0(c_{t^*}, \mathcal{L}_{c_{t^*}}) \tag{4.37}$$

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as the number of vanishing eigenvalues of ι_1, i.e. the algebraic multiplicity of the eigenvalue 0, is $\geq$ its geometric multiplicity. So, if $h^0(c_t, \mathcal{L}_{c_t}) = m$ generically for $f(t) = 0$ then (if - as is usually the case and we assume - $f \neq g^k$ with $k > 1$, i.e. $\mathrm{ord}_t f = 1$) one has $f^m | Pfaff$ (as $\mathrm{ord}_t \det \iota_1 = \mathrm{ord}_t f \cdot \mathrm{mult}_f \det \iota_1$) and one has for the precise multiplicity

$$k' := \mathrm{mult}_f Pfaff \geq k_{(f)} := h^0(c_t, \mathcal{L}_{c_t})|_{f(t)=0,\, generic} \tag{4.38}$$

As $h^0(c_t, \mathcal{L}_{c_t})$ is upper semicontinuous we know also the following implication

$$\Sigma \subset (f) \Longrightarrow k_\Sigma \geq k_{(f)} \tag{4.39}$$

To infer from (4.38) and $k_{(f)}$ a multiplicity $k' = \mathrm{mult}_f\, Pfaff > 1$ (i.e. an interesting *lower* bound) one would need however an opposite inequality, say a *lower* bound on $k_{(f)}$ in (4.39). We will get an *upper* bound for k' from consideration of degrees, cf. (4.59).

The multiplicity along Σ. We are particularly interested in the multiplicity $h^0(c_t, \mathcal{L}_{c_t})$ along the locus Σ (which can also have higher codimension) for which effective bounds can be derived. The proof in (4.19) of $\Sigma \subset (Pfaff)$ used a representation of $\mathcal{L}_{c_t}$ for $t \in \Sigma$ as $\mathcal{O}_{c_t}(D)$ with D effective (where of course $\deg D = \deg K_c^{1/2} = g_c - 1$). So, by Cliffords theorem,

$$t \in \Sigma \quad \Longrightarrow \quad h^0(c_t, \mathcal{L}_{c_t}) \leq \frac{n}{2}\left(r - \frac{n+1}{2}\right) + 1 \tag{4.40}$$

On the other hand the specialisations in (4.19) allow also to derive a lower bound of $h^0(c_t, \mathcal{L}_{c_t})$ for $t \in \Sigma$. In both cases one finds by consideration of the long exact sequence that the sections of the corresponding bundle on $\mathcal{E}$ in (4.19) *inject* into the corresponding sections of the bundle restricted to c_t. This gives the following estimates ($m := r - \frac{n+1}{2}$)

$$t \in \Sigma \quad \Longrightarrow \quad h^0(c_t, \mathcal{L}_{c_t}) \geq \begin{cases} \frac{1}{2}\frac{n}{2}(r - \frac{n}{2}) + 1 & \text{for } \; n \equiv 0\,(2) \\ (r - \frac{n+1}{2}) + 1 & \text{for } \; n \not\equiv 0\,(2) \end{cases} \tag{4.41}$$

where we evaluated $h^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(\frac{n}{2}s + \frac{r-1}{2}F)) = h^0(b, \mathcal{O}_b(\frac{r-1}{2}) \oplus \bigoplus_{i=2}^{n/2} \mathcal{O}_b(\frac{r-1}{2} - i)) = \frac{r-1}{2} + 1 + \sum_{i=2}^{n/2}(\frac{r-1}{2} - i + 1) = \frac{n}{2}(\frac{r-1}{2} + 1) - (\frac{1}{2}\frac{n}{2}(\frac{n}{2} + 1) - 1)$ in the first line (note $r > n$); so

$$\frac{1}{2}\frac{n}{2}\left(m + \frac{1}{2}\right) \leq h^0(c_t, \mathcal{L}_{c_t}) - 1 \leq \frac{n}{2}m \qquad \text{for } \; n \equiv 0\,(2) \tag{4.42}$$

$$m \leq h^0(c_t, \mathcal{L}_{c_t}) - 1 \leq \frac{n}{2}m \qquad \text{for } \; n \not\equiv 0\,(2) \tag{4.43}$$

4.6 The case of SU(3) bundles

4.6.1 The relation $\mathcal{R} = \Sigma$ for SU(3) bundles

Here one has from the spectral equation

$$C_r z + B_{r-2} x + A_{r-3} y = 0 \tag{4.44}$$

that[17] $z \in H^0(c, \mathcal{O}_{\mathcal{E}}(3s)|_c)$ has the vanishing divisor (where the u_j are the zeroes of A_{r-3})

$$(z) = 3s|_c = \sum_{j=1}^{r-3} \{3p_0, u_j\} \tag{4.45}$$

Now let $t \in \Sigma$ and let $(0 \neq)\zeta \in H^0(c_t, \mathcal{O}_{\mathcal{E}}(3s - \sum_{j=1}^{r-3} F_{u_j})|_{c_t})$. As the flat bundle of which ζ is a nontrivial section must be trivial ζ remains everywhere nonvanishing. Therefore[18] z/ζ is not only an element of $\Gamma_{\text{mero}}(c, \mathcal{O}_{\mathcal{E}}(\sum_{j=1}^{r-3} F_{u_j})|_{c_t})$ but one has even

$$\frac{z}{\zeta} \in H^0\Big(c, \mathcal{O}_{\mathcal{E}}\Big(\sum_{j=1}^{r-3} F_{u_j}\Big)\Big|_{c_t}\Big) \tag{4.46}$$

Now tensoring the short exact sequence $0 \longrightarrow \mathcal{O}_{\mathcal{E}}(-c_t) \longrightarrow \mathcal{O}_{\mathcal{E}} \longrightarrow \mathcal{O}_{c_t} \longrightarrow 0$ with $\mathcal{O}_{\mathcal{E}}((r-3)F)$ and taking the long exact cohomology sequence tells one that actually

$$H^0\Big(c, \mathcal{O}_{\mathcal{E}}\Big(\sum_{j=1}^{r-3} F_{u_j}\Big)\Big|_{c_t}\Big) = \pi^* H^0\Big(b, \mathcal{O}_b(r-3)\Big) \tag{4.47}$$

because $H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(-3s-3F)) = 0$, cf. equ.'s (C.22) and (C.17) of [9]. Therefore one gets that $z/\zeta|_{c_t} = \pi^* P_{r-3}|_{c_t}$ and thus (with the u'_k denoting the $r-3$ zeroes of P_{r-3})

$$\sum_{k=1}^{r-3} F_{u'_k}|_{c_t} = (\pi^* P_{r-3})|_{c_t} = \Big(\frac{z}{\zeta}\Big)\Big|_{c_t} = 3s|_{c_t} \tag{4.48}$$

such that, whereas (4.32) and (4.33) gave $\mathcal{R} \subset \Sigma$, one gets here also that $\Sigma \subset \mathcal{R}$ (as $\mathcal{R}$ is, cf. (4.33), the locus of $t \in \mathcal{M}_{\mathcal{E}}(c)$ where $3s|_{c_t}$ equals the sum of the $r-3$ fibers $F_{u_j}|_{c_t}$ of $\pi_{c_t} : c_t \longrightarrow b$ over the zeroes of A_{r-3}; now the fibers $F_{u'_k}|_{c_t}$ in (4.48) must be necessarily these as they must contain the $\{p_0, u_j\}$). So for SU(3) bundles we find finally that

$$\mathcal{R} = \Sigma \tag{4.49}$$

In general, for SU(3) bundles one has for the multiplicity $k_\Sigma = h^0(c_t, \mathcal{L}_{c_t})$ along Σ

$$r - 1 \leq k_\Sigma \leq \frac{3}{2} r - 2 \tag{4.50}$$

Let us recall the locus $\mathcal{R}$ (we switch between both notations: $a_2 = B_{r-2}$, $a_3 = A_{r-3}$)

$$\mathcal{R} = \bigcap_{j=1}^{r-3} (R_2^{(j)}) = \bigcap_{j=1}^{r-3} (Res(a_2, a_3^{(j)})) \tag{4.51}$$

of codim $r-3$, i.e. the locus where $A_{r-3}|B_{r-2}$ or

$$\Sigma = \mathcal{R} = \Big\{ t \in \mathcal{M}_{\mathcal{E}}(c) \Big| \exists L_1 \; s.t. \; B_{r-2} = L_1 A_{r-3} \Big\} \tag{4.52}$$

Here codim $\Sigma = r - 3$, so $= 1, 2$ for $r = 4, 5$, resp. (cf. (4.62), (4.63) and Ex.'s 1, 2 [9]).

Here the locus R, cf. appendix A.4, is a $\leq$ codim 1 subspace (cf. section 3.2.2 of [9]) where $(3s - F)|_{c_t} \sim$ effective holds; the latter by R-R holds always for $r \geq 6$ and for $r = 5$ (or $r = 4, \lambda > \frac{3}{2}$, cf. after (A.21)) poses a condition.

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[17] We will not make any notational distinction between the section z over $\mathcal{E}$ and its restriction to c_t.

[18] Recall that a holomorphic/meromorphic section is a collection of holom./merom. functions transforming with suitable transition functions (and a product of sections is a section of the product bundle).

4.6.2 SU(3) bundles with $\lambda = 3/2$ and the special locus (f)

Let us describe one important special case in a more detailed manner. For $\lambda = 3/2$ one has $l(-F) = \mathcal{O}_{\mathcal{E}}(6s - (r-4)F)$ and we took, cf. [9],

$$\bar{l}(-F) = \begin{cases} \mathcal{O}_{\mathcal{E}}(3s - \frac{r-4}{2}F) & \text{for } r \text{ even} \\ \mathcal{O}_{\mathcal{E}}(3s - \frac{r-3}{2}F) & \text{for } r \text{ odd} \end{cases} \tag{4.53}$$

We consider $\iota_1 : H^1(\mathcal{E}, \widetilde{\mathcal{L}}(-c)) \to H^1(\mathcal{E}, \widetilde{\mathcal{L}})$ with $\widetilde{\mathcal{L}} = l(-F)$ and the corresponding map $\bar{\iota}_1$ for $\bar{l}(-F)$. We got $f|Pfaff$ for

$$Pfaff = \det \iota_1 \tag{4.54}$$

$$f = \det \bar{\iota}_1 \tag{4.55}$$

For r even one has $f \equiv 0$ as $h^0(c, \bar{l}(-F)|_c) = -3(\frac{r}{2} - 1) + h^0(c, \mathcal{O}_{\mathcal{E}}(3(\frac{r}{2} - 1)F)|_c) \geq 1$, such that also $Pfaff \equiv 0$ as $h^0(c, l(-F)|_c) \geq 1$ by $l(-F)|_c = \bar{l}(-F)|_c^{\otimes 2}$. For r odd the matrix representation (G.12) of [9] for $\bar{\iota}_1$ shows (via linear dependence of the rows)

$$(f) = \left\{ t \in \mathcal{M}_{\mathcal{E}}(c) \middle| \, \exists\, \tilde{C}_k, \tilde{B}_{k+2}, \tilde{A}_{k+3}, k = \tfrac{r-7}{2}, s.t. \;\; \tilde{C}C_r + \tilde{B}B_{r-2} + \tilde{A}A_{r-3} = 0 \right\} \tag{4.56}$$

(where $\tilde{C} = 0, (f) = R$ for $r = 5$). So one gets from (4.52) (cf. also (4.60), (4.62) below)

$$\Sigma = \mathcal{R} \subset (f) \subset (Pfaff) \tag{4.57}$$

(by using $\tilde{C} = 0, \tilde{B} \equiv 1, \tilde{A} = -L_1$). If (for r odd) now $t \in (f)$, i.e. if a nonzero element $\rho \in H^0(c, \bar{\mathcal{L}}_t)$ exists, one gets linearly independent elements $\rho^2 u, \rho^2 v \in H^0(c, \mathcal{L}_t)$, so

$$k_{(f)} \geq 2 \tag{4.58}$$

4.6.3 More specialised information for certain SU(3) bundles

In [9] we looked, following [10], for components $k'(f)$, where $k' = \text{mult}_f\, Pfaff$, of the reducible vanishing divisor $(Pfaff) = (\det \iota_1)$ of the Pfaffian. The factor f (respectively $Res(a_2, a_3)$ in Example 2) arose as a $\det \bar{\iota}_1$ for a related line bundle $\bar{l}(-F)$. We get

$$k'\, h^1\big(\mathcal{E}, \bar{l}(-F)\big) \leq h^1\big(\mathcal{E}, l(-F)\big) \tag{4.59}$$

as an upper bound on k' from consideration of degrees and in [9] we got (for $\chi = 1$) the following results[19] (subscripts in brackets indicate the polynomial degree):

(A) $\underline{r \geq 5 \text{ odd}, \lambda = 3/2}$ with $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(6s - (r-4)F)|_c$ and (with f from (4.53))

$$(Pfaff_{(6r-10)}) = k'(f_{(\frac{3r-5}{2})}) + (g) \tag{4.60}$$

Here $4 \geq k' \geq k_{(f)} \geq 2$ from consideration of degrees, (4.38) and (4.58) (and $k_\Sigma \geq k_{(f)}$ with $k_\Sigma \geq 4$ by $\Sigma \subset (f)$, (4.39) and (4.50)), so predicting $k' \geq 2$.

[19] Besides an SU(4) case of $r = 9, \lambda = 1$ with $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(6s - F)|_c$ and $(Pfaff_{(20)}) = (f_{(9)}) + (g_{(11)})$ (where $\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(4s - F)$).

(B) $r = 5, \lambda = 5/2$ with $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(9s - 3F)|_c$ and (with $f = \det \bar{\iota}_1 = Res(B_3, A_2)$ where $\bar{l}(-F) = \mathcal{O}_{\mathcal{E}}(3s - F)$ and $k' \leq 12$ from degrees while $k_\Sigma = 4$ or 5)

$$(Pfaff_{(62)}) = k'(f_{(5)}) + (g) \tag{4.61}$$

We considered, with the notation $\Sigma = (f_\Lambda)$, in greater detail the following cases [9]:

- Example 1: $r = 4, \lambda = 5/2$ with $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(9s - F)|_c$ and (with R the zero-divisor of the polynomial $R_2 = Res(a_3, a_2)$ which is of degree 3) (cf. also appendix A.4)

$$(Pfaff_{(44)}) = k' R_{(3)} + (Q_{(44-3k')}) \tag{4.62}$$

  Here $(R_2) = R = \mathcal{R} = \Sigma$ and $\operatorname{codim} \mathcal{R} = \operatorname{codim} \Sigma = 1$ from (4.34), so Σ is a component of $(Pfaff)$. As here $\Lambda = \mathcal{O}_{\mathcal{E}}(3s - F)$ one gets that $\mathcal{L} \cong \mathcal{O}_{\mathcal{E}}(2F)|_c$ along Σ such that $3 \leq k_\Sigma \leq 4$, predicting $k' \geq 3$ by (4.38) (with $\Sigma = (R_2)$) in accord with $k' = 11$, cf. [10].

- Example 2: $r = 5, \lambda = 3/2$ (a special case of (A) above) with $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(6s - F)|_c$ and

$$(Pfaff_{(20)}) = k' R_{(5)} + (g_{(20-5k')}) \tag{4.63}$$

  where $R = (R_2) = (Res(a_2, a_3))$ such that $(3s - F)|_{c_t} \sim$ effective along R (cf. appendix A.4); for $r = 5$ this is enough for $(6s - (r-4)F)|_{c_t} \sim$ effective, i.e. $f|Pfaff$ or $R = (f) \subset (Pfaff)$; but one has not yet $(3s - (r-3)F)|_{c_t} \sim$ effective for $\Lambda = \mathcal{O}_{\mathcal{E}}(3s - (r-3)F)$, which needs for $r = 5$ a second condition to be posed (so $\operatorname{codim} \Sigma = 2$). Here one has $4 \geq k' \geq k_R \geq 2$ from consideration of degrees, (4.38) and (4.58) (and $k_\Sigma \geq k_R$ with $4 \leq k_\Sigma \leq 5$ by $\Sigma = \mathcal{R} \subset R$, (4.39) and (4.50)), so predicting $k' \geq 2$ and concretely $k' = 4$, cf. [10].

5 The moduli space over the instanton curve

The contribution criterion (1.5), and even the precise prefactor, depends only on the restriction $V|_b$ of V to the instanton curve b, i.e. the Pfaffian factors through the map

$$\mathcal{M}_X(V) \xrightarrow{\rho_b} \mathcal{M}_b(V_b) \tag{5.1}$$

(defined by $V \to V|_b$) to the moduli space of $\mathrm{SU}(n)$ bundles V_b over b. By Grothendieck's decomposition theorem one has (the array of the k_i is called the splitting type)

$$V_b = \bigoplus_{i=1}^{n} \mathcal{O}_b(k_i) \tag{5.2}$$

This concerns $Gl(n, \mathbf{C})$ or, equivalently, $\mathrm{U}(n)$ bundles. For $\mathrm{SU}(n)$ bundles we get the further condition $\int_b c_1(V|_b) = \sum_{i=1}^n k_i = 0$, giving the space of sets of $n-1$ integers as moduli space: considering the lattice $\Gamma_{\mathrm{SU}(n)}$

$$0 \longrightarrow \Gamma_{\mathrm{SU}(n)} \longrightarrow \mathbf{Z}^n \xrightarrow{\Sigma} \mathbf{Z} \longrightarrow 0 \tag{5.3}$$

with the natural symmetry action of $W_{\mathrm{SU}(n)}$, the symmetric group in $n-1$ elements, gives the moduli space (the complement of the divisor $(Pfaff)$ in $\mathcal{M}_X(V)$ is $\rho_b^{-1}\big((0,\ldots,0)\big)$)

$$\mathcal{M}_b(V_b) = \Gamma_{\mathrm{SU}(n)}/W_{\mathrm{SU}(n)} \tag{5.4}$$

A world-sheet instanton supported on b contributes according to the criterion [4]

$$W_b \neq 0 \iff h^0\big(b, V|_b(-1)\big) = 0 \tag{5.5}$$

with $V|_b(-1) := V|_b \otimes \mathcal{O}_b(-1)$ and $h := h^0\big(b, V|_b(-1)\big) = \sum_{k_i>0} k_i$. So $h=0 \Leftrightarrow k_i = 0$, for all i, i.e. b contributes precisely if $V|_b$ is trivial: $W_b \neq 0 \iff V|_b = \bigoplus_1^n \mathcal{O}_b$; and $h = h^0\big(b, \pi_{c*} l|_c \otimes \mathcal{O}_b(-1)\big) = h^0(c,\mathcal{L})$. One has[20] $\mathcal{L} = K_c^{1/2} \otimes \mathcal{F}$, with the flat $\mathcal{F} = \Lambda|_c^\lambda$ for n even; for n odd the decomposition depends on a spin choice with its Pic_0^2 ambiguity. (For SU(2) bundles $V|_b = \mathcal{O}_b(h) \oplus \mathcal{O}_b(-h)$ with $h := h^0(c,\mathcal{L}) = h^0(b, V|_b(-1))$.[21])

By (4.17), (4.18) one gets, in $\mathcal{M}_\mathcal{E}(c)$, on *the complement of* $(Pfaff)$ (contained in the complement of Σ) that $k_i = -k_i$ for all i, and on $\Sigma_{2\lambda}(\supset \Sigma)$ that $\{k_i\} = \{-k_i\}$ as sets, i.e. there one has

$$V|_b = \begin{cases} \bigoplus_{i=1}^{n/2} \mathcal{O}_b(k_i) \oplus \mathcal{O}_b(-k_i) & \text{for } \ n \equiv 0 \ (\,mod\, 2) \\ \mathcal{O}_b \oplus \bigoplus_{i=1}^{(n-1)/2} \mathcal{O}_b(k_i) \oplus \mathcal{O}_b(-k_i) & \text{for } \ n \not\equiv 0 \ (\,mod\, 2) \end{cases} \tag{5.6}$$

$(Pfaff) \subset \mathcal{M}_\mathcal{E}(c)$, the set where a $k_i \neq 0$ exists, contains Σ (where $\{k_i\} = \{-k_i\}$).

6 Intrinsic derivation of $Pfaff$ on the instanton curve

In [9] we stated the following necessary criterion for contribution to the superpotential

$$W_b \neq 0 \implies \beta < 0 \tag{6.1}$$

(cf. equ. (4.27) of [9]); recall the notation $\widetilde{\mathcal{L}} = l(-F) = \mathcal{O}_\mathcal{E}(n(\lambda + \frac{1}{2})s + \beta F) =: \mathcal{O}_\mathcal{E}(\widetilde{D})$. This was done via an extrinsic detour over the sheafs $\widetilde{\mathcal{L}} \otimes \mathcal{O}_\mathcal{E}(-c)$ and $\widetilde{\mathcal{L}}$ on $\mathcal{E}$. Actually there is also a more direct, intrinsic argument, using directly the sheaf $\mathcal{L} = \widetilde{\mathcal{L}}|_c$ on c.

Recall that $h^0(c, \mathcal{O}_c(D)) - 1$ is the (projective) dimension of the associated linear system $|D|$ (of effective divisors linear equivalent to D) of a divisor D on c (with $\mathcal{O}_c(D)$ here as the associated line bundle on c). So one recovers immediately (6.1) by (cf. (4.35) of [9])

$$h^0(c, \mathcal{O}_c(D)) = 0 \iff D \not\sim \text{ effective} \implies D \neq \text{ effective} \tag{6.2}$$

as $W_b \neq 0$ means according to the precise criterion (5.5) just $h^0(c,\mathcal{L}) = 0$.

One has the representation $\mathcal{L} = K_c^{1/2} \otimes \mathcal{F} = \mathcal{O}_c(D) = \mathcal{O}_\mathcal{E}(\widetilde{D})|_c$ of the relevant line bundle as being composed[22] of the spin bundle of c and a flat bundle with $d := \deg D = g-1$

[20] Note $h^0\big(b, V|_b(-1)\big) - h^1\big(b, V|_b(-1)\big) = \int_b c_1\big(V|_b(-1)\big) + \frac{c_1(b)}{2} = \int_b c_1(V)|_b = 0$ such that $0 = \deg V|_b = \chi(b, V|_b(-1)) = \chi(c, l(-F)|_c) = \deg l(-F)|_c - \deg K_c^{1/2}$ (even $h^i(b, V|_b(-1)) = h^i(c, l(-F)|_c)$).

[21] The jumping phenomenon arises if a one-parameter family of spectral curves $c' \in |c|$, with a corresponding family of line bundles $\widetilde{\mathcal{L}}|_{c'} \in Pic_{g-1}(c')$, meets $\Theta_{c'}$ at $t=0$, say at a generic smooth point: then $V|_b$ is $\mathcal{O}_b \oplus \mathcal{O}_b$ for $t \neq 0$ and $\mathcal{O}_b(h) \oplus \mathcal{O}_b(-h)$ at $t=0$ where $h=1$.

[22] This is for n even; for n odd a corresponding decomposition depends on a spin choice with its Pic_0^2 ambiguity; the assertion, and what follows, then has to be somewhat adjusted in notation, cf. discussion after (4.7), (4.8) and footn. 16, as the factors of $\mathcal{L}$ then do not arise by restriction of line bundles on $\mathcal{E}$.

and $g := g_c$ and $\mathcal{F}_t = \mathcal{O}_{c_t}(G|_{c_t})$ with $G = \lambda(ns - (r - n\chi)F)$ for a V_t with $t \in \mathcal{M}_{\mathcal{E}}(c)$. Recall the map[23] $D \to \mu(D) = \sum_i \mu(p_i) = (\sum_i \int_{p_0}^{p_i} \omega_\alpha)_{\alpha=1,\dots,g}$ (the point p_0 chosen fixed) first from effective divisors $D = \sum_{i=1}^d p_i$ (then prolonged to all divisors) to the Jacobian

$$\mu : \; Sym^{g-1} c \;\longrightarrow\; Jac(c) \tag{6.3}$$

of image $W_{g-1} = \Theta - \kappa$ with $-\kappa = \mu(\frac{1}{2}K_c)$ by a theorem of Riemann (with $W_d := \mu(Sym^d c)$ and $\frac{1}{2}K_c$ understood as divisor class); enhancing μ to a map from all divisors (cf. appendix B.2) one finds that the preimage of W_{g-1} are those divisors of degree $g-1$ which are $\sim$ effective. Then one has, as $W_b \neq 0 \Longleftrightarrow \frac{1}{2}K_c + G|_c \not\sim$ effective by (5.5) and (6.2),

$$Pfaff\,(\bar{\partial}_{V_t|_b(-1)}) = 0 \Longleftrightarrow \theta\big(\mu(G|_{c_t}), \Omega_t\big) = 0 \tag{6.4}$$

(cf. (6.5)). The fact that θ is an even function $\theta(-z, \Omega) = \theta(z, \Omega)$ corresponds to the fact that $\mathcal{L}_{-\lambda}$ contributes exactly if $\mathcal{L}_\lambda$ does as $h^0(c, \mathcal{L}_\lambda) = h^0(c, \mathcal{L}_{-\lambda})$, cf. equ. (E.3) in [9].

6.1 Relation of the extrinsic and intrinsic considerations

We consider the bundle $\underline{Jac}$ of Jacobians over the moduli space $\mathcal{M}_g$, cf. appendix B.2. In each fibre $Jac(c_\Omega)$ one has the respective theta divisor Θ_Ω of $\theta(\cdot, \Omega)$. These divisors fit together to a divisor Θ in the total space $\underline{Jac}$. In our concrete situation the extrinsic moduli space is embedded (for Π an injective immersion) in the base, via a map Π

$$\mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)) \;\ni\; t \;\stackrel{\Pi}{\longrightarrow}\; \Pi(t) \;=\; \Omega_t \;\in\; \mathcal{M}_g \tag{6.5}$$

with image I, say. Here t stands for the specific, concrete curve $c_t \in |c|$ inside $\mathcal{E}$ (which is parametrized by t in the moduli space) given by the zero set (s_t) of a section s_t in $H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$. Now let us enhance the perspective to the respective full fibrations

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{E}}(V_{\mathcal{E}}) & \longrightarrow & \underline{Jac} \\ \downarrow \; Jac(\cdot) & & p \downarrow \; Jac(\cdot) \\ \mathbf{P}H^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big) & \stackrel{\Pi}{\longrightarrow} & \mathcal{M}_g \end{array} \tag{6.6}$$

Using the additional information provided by the line bundle $\widetilde{\mathcal{L}}|_{c_t}$ over c_t one gets (λ held fixed) a section[24] $\mathcal{S}_\lambda$ over $I = im\,\Pi \subset \mathcal{M}_g$ of the $\underline{Jac}$-fibration from the section one started with in the fibered space $\mathcal{M}_{\mathcal{E}}(V_{\mathcal{E}})$ (which itself came from the original V over X)

$$\Gamma\big(\mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)), \mathcal{M}_{\mathcal{E}}(V|_{\mathcal{E}})\big) \;\ni\; S_\lambda \;\;\longrightarrow\;\; \mathcal{S}_\lambda \;\in\; \Gamma\big(I, \underline{Jac}\big) \tag{6.7}$$

[23] For ω_α a basis of holomorphic differentials normalized by $\int_{a_\alpha} \omega_\beta = \delta_{\alpha\beta}$ with respect to a canonical symplectic basis $(a_1, \dots, a_g, b_1, \dots, b_g)$ of $H^1(c, \mathbf{Z})$, i.e. $a_\alpha \cdot a_\beta = b_\alpha \cdot b_\beta = 0$, $a_\alpha \cdot b_\beta = \delta_{\alpha\beta}$; recall $Jac(c) = \mathbf{C^g}/\Lambda$ with the lattice Λ generated by the a- and b- periods of the ω_α; μ is still well-defined on divisor classes, cf. appendix B where the Riemann theta-function θ and its divisor Θ are recalled.

[24] Note that, contrary to a situation where one has a family of linear equivalent divisors on one fixed curve which are mapped to just one point in a Jacobian, here one has one divisor $\mathcal{D}$ (in the ambient space $\mathcal{E}$, with $\widetilde{\mathcal{L}} = \mathcal{O}_{\mathcal{E}}(\mathcal{D})$) which restricts to divisors on the respective members of a family $|c|$ of linear equivalent curves; the ensuing family of divisors is then mapped to Jacobian fibre points over the respective base points in $I \subset \mathcal{M}_g$ (themselves corresponding to the different members in $|c|$).

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(a global spin choice made); so this continues the map of fibrations in (4.4). In other words, one has a (choice-dependent) mapping[25] from $t \in \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$ to corresponding pairs $(z(\Omega_t), \Omega_t)$ where $\Omega_t = \Pi(t) \in I \subset \mathcal{M}_g$ and $z(\Omega_t) = \mu(G_{c_t}) \in Jac(c_t)$ (n even, say)

$$\begin{aligned} \mathcal{M}_{\mathcal{E}}(V_{\mathcal{E}}) \quad &\ni \quad (S_\lambda(t)\,,\, t) \quad = \quad (\mathcal{L}_{c_t}, t) \\ &\longrightarrow (\mathcal{S}_\lambda(\Omega_t), \Omega_t) = (\mu(G_{c_t}), \Omega_t) = ([z_t], \Omega_t) \;\in\; \underline{Jac} \end{aligned} \tag{6.8}$$

Considering just the part $\underline{Jac}|_I$ and the corresponding restricted global theta divisor $\Theta|_I$, the question is where (i.e. at which points of I) the section $\mathcal{S}_\lambda$ meets that divisor

$$p(\mathcal{S}_\lambda \cap \Theta|_I) \tag{6.9}$$

One expects therefore an identification of the following divisors (with $z(\Omega)$ as described)

$$\mathcal{H} = \big\{t \,\big|\, \det\ \iota_1(t) = 0\big\} \subset \mathbf{P}H^0\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c)\big) \xrightarrow{\ \Pi\ } \big\{\Omega \in I \,\big|\, \theta\big(z(\Omega), \Omega\big) = 0\big\} \subset I \subset \mathcal{M}_g \tag{6.10}$$

The fibre has dimension g_c. But the whole space of vertical moduli in each respective fibre $Jac(c_t)$, i.e. of all flat line bundles on c_t, is not at our disposal as arguments for $Pfaff$; rather, the available variation stems just from a movement in the base I (which comes from motions in $|c|$, i.e. motions of c in $\mathcal{E}$, that is moduli of the restricted bundle); in each fibre only the section point of $\mathcal{S}_\lambda$ occurs as argument of $Pfaff$. Therefore, whereas an a priori *intrinsic* investigation would be concerned with a divisor condition in the full *vertical* (Jacobian) direction, *extrinsically* one investigates the condition inside $|c|$, respectively its image I in the base $\mathcal{M}_g$, i.e. along a *horizontal* (base) direction. One expects by (6.10) even the identification of a transcendental function as a polynomial

$$\det\ \iota_1\,(t) \quad = \quad \theta\big(\mathcal{S}_\lambda\big(\Pi(t)\big), \Pi(t)\big) \tag{6.11}$$

with the θ-function (here actually $\mathcal{S}_\lambda\big(\Pi(t)\big)$ gives only $[z(\Omega_t)] \in Jac(c_t)$) and not $z(\Omega_t)$)

$$\theta\big(\mathcal{S}_\lambda\big(\Pi(t)\big), \Pi(t)\big) \;=\; \sum_{l \in \mathbf{Z}^{\mathbf{g}}} \exp\left\{2\pi i\left(\frac{1}{2}\Big\langle l, \Pi(t) l\Big\rangle + \Big\langle l, \mathcal{S}_\lambda\big(\Pi(t)\big)\Big\rangle\right)\right\} \tag{6.12}$$

where we recall that[26] (for explicitness we describe the case n even where $\mathcal{F}_{c_t} = \mathcal{O}_{c_t}(G|_{c_t})$, so that we do not discuss integrality-issues and 2^{2g}-ambiguity, cf. footn. 22)

$$\Pi(t) = \Omega_t \in \mathcal{H}_g \qquad \text{with} \quad \Omega_t \;=\; \left(\int_{b_\beta} \omega_\alpha\right) \tag{6.13}$$

$$\mathcal{S}_\lambda\Big(\Pi(t)\Big) = [z_t] \in \mathbf{C}^{\mathbf{g}}/\Lambda \quad \text{with} \quad [z_t] \;=\; \mu(G|_{c_t}) = \left(\sum_{i=1}^{\lambda n(r-n)} \int_{q_i}^{p_i} \omega_\alpha\right)_{\alpha=1,\dots,g} \tag{6.14}$$

[25]Here we give the fibre coordinates first as is customary in connection with the theta function $\theta(z, \Omega)$ which itself is defined on the set of $(z, \Omega_t) \in \underline{Jac}$. Furthermore, by abuse of language, we use the same symbol for a section and the function giving the fibre coordinate in dependence of the base point.

[26]With (a_i, b_i), $i = 1, \dots, g$, a canonical symplectic basis of $H_1(c_t, \mathbf{Z})$, 'constant' under varying t.

Here the ω_α, $\alpha = 1, \ldots, g$ are a basis of $H^{1,0}(c_t)$ (cf. remark after (4.12) how they arise concretely by taking Poincare residues), normalized by $\int_{a_\beta} \omega_\alpha = \delta_{\alpha\beta}$. Furthermore one has (where $a_n(u_i) = 0$, cf. appendix 4.4)

$$G|_{c_t} = \lambda n s|_{c_t} - \lambda(r-n)F|_{c_t} = \sum_{i=1}^{\lambda n(r-n)} (p_i - q_i) \quad \text{where} \begin{cases} s|_c = \sum_{i=1}^{r-n} \{p_0, u_i\} \\ F_{u_*}|_c = \sum_{i=1}^{n} \{q_i, u_*\} \end{cases} \tag{6.15}$$

and where z_t is only well-defined up to the periods, i.e. as $[z_t] \in Jac(c_t) \cong \mathbf{C}^{\mathbf{g}}/\Lambda$. Here z_t arises from an object G on $\mathcal{E}$ which remains constant for a varying $t \in \mathcal{M}_{\mathcal{E}}(c)$.

As Σ denotes the subset of $\mathcal{M}_{\mathcal{E}}(c)$ where $(ns - (r-n)F)|_{c_t} \sim 0$ one has also (cf. (4.18))

$$\Sigma \subset \{Pfaff(\cdot) = 0\} \longrightarrow \mathcal{S}_\lambda(\Pi(\Sigma)) \subset \Theta \tag{6.16}$$

6.2 Some further remarks about multiplicities

We conclude these comparisons by continuing the considerations in section 4.5: we now want to compare the perspectives which the extrinsic and the intrinsic approach offer, respectively, on the issue of multiplicity.

6.2.1 Extrinsic point of view

In the extrinsic consideration one has a moduli space $\mathcal{M}_{\mathcal{E}}(c) = |c| = \mathbf{P}H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(c))$ and a line bundle $\widetilde{\mathcal{L}}|_{c_t}$ over each of the respective curves $c_t \in |c|$. Inside $|c|$ one has a hypersurface $\mathcal{H} = \{\det \iota_1 = 0\}$ of degree $h^1(\mathcal{E}, \widetilde{\mathcal{L}})$. The matrix for the map

$$\iota_1 : H^1(\mathcal{E}, \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{E}}(-c)) \longrightarrow H^1(\mathcal{E}, \widetilde{\mathcal{L}}) \tag{6.17}$$

arises from multiplication with the defining polynomial s_t of the curve $c_t = \{s_t = 0\} \in |c|$. Therefore the moduli, which are essentially given by the coefficients of the monomials in s, enter the matrix-elements linearly. So $\det \iota_1 = 0$ is one equation (of determinantal form) of degree $h^1(\mathcal{E}, \widetilde{\mathcal{L}})$ in the moduli. That is, in the extrinsic approach of equ. (1.1), the Pfaffian is identifed as a polynomial in the moduli which has the form of a determinant. From the determinantal nature of $Pfaff$, with respect to *horizontal* variations, i.e. variations of the specific curve c_t (which means varying the bundle V_T over b, where $T \in \mathcal{M}_X(V)$ with $T \longrightarrow t \longrightarrow z_t$), one finds then the following. One has the number $h^0(b, S_-(b) \otimes V|_b)$ of occuring zero-modes, so one gets the refinement of the contribution criterion (5.5) that $Pfaff$ vanishes (at least) to order $h^0(b, S_-(b) \otimes V_T|_b)$, that is one has

$$ord_t\, Pfaff \geq h^0(c_t, \mathcal{L}_{c_t}) \tag{6.18}$$

(cf. equ. (4.37)). We wish to emphasize that here the source domain of values for $Pfaff$ is a horizontal one, in the sense described. This is typical for the extrinsic algebraic approach. In the intrinsic transcendental approach, one considers the general theta function $\theta(z, \Omega_c)$ of two variables and is usually concerned with its variation in the first entry, the second one held fixed. The latter, describing the isomorphism type of the concrete curve c, corresponds to the mentioned horizontal variations of the concrete curve $c_t \in \mathcal{M}_{\mathcal{E}}(c)$ in the extrinsic approach. By contrast the variable z corresponds under the Abel/Jacobi-map μ to specific divisor classes, or equivalently line bundles, on a fixed curve (this is what we call here the intrinsic aspect of the transcendental approach). Despite this difference it is interesting to see nevertheless a corresponding role of $h^0(c_t, \mathcal{L}_{c_t})$ for $\theta(\cdot, \Omega_c)$ in the intrinsic approach.

6.2.2 Intrinsic point of view

If $z = \mu(G_{c_t}) \in Jac(c)$ corresponds to $t \in \mathcal{M}_{\mathcal{E}}(V|_{\mathcal{E}})$ we saw in (6.4) that $\theta(z) = 0 \Leftrightarrow Pfaff(t) = 0$ as Riemann's theorem $W_{g-1} = \mu(K_c^{1/2}) + \Theta_c$ shows that just then an *effective* divisor class of degree $g-1$ belongs to the line bundle $\mathcal{L} = K_c^{1/2} \otimes \mathcal{F}_z$ (such that $S_-(c) \otimes \mathcal{F}$ has non-trivial sections and b can not contribute to the superpotential).

Actually Riemann did not stop at the relation described so far but elucidated (in terms of the associated linears systems on the curve) the finer structure of the analytic subvariety given by the theta-divisor, especially its stratification by singular points.

We saw that for $\mu(G|_c) \in \Theta$ the divisor class $\frac{1}{2}K_c + G|_c$ related to $\mathcal{L}$ was effective, so

$$z = \mu(\mathcal{L}) - \mu(K_c^{1/2}) \ \in \ \Theta \Longrightarrow h^0(c, \mathcal{L}) \ > \ 0 \tag{6.19}$$

Now recall first the characterisation of the regular locus

$$\mu(D_d) \ \in \ W_d \ \text{ is a smooth point} \Longleftrightarrow h^0(c, D_d) \ = \ 1 \tag{6.20}$$

This gives in our case the following special case of Riemann's singularity theorem

$$z = \mu(\mathcal{L}) - \mu(K_c^{1/2}) \ \in \ \Theta_{\text{reg}} \Longleftrightarrow h^0(c, \mathcal{L}) \ = \ 1 \tag{6.21}$$

Riemann's theorem asserts (where codim $\Theta_{\text{sing}} = 3$ if c is not hyperelliptic; then it is 2)

$$\text{mult}_z\, \Theta = h^0(c, \mathcal{L}) \tag{6.22}$$

Here one considers the $\theta(z, \Omega)$-function for fixed argument Ω, i.e. one considers in the Jacobian fibration (B.9) over $\mathcal{M}_g$ just the *vertical* direction.

From $\theta(z) = 0 \Longleftrightarrow Pfaff(t) = 0$ above one gets that $Pfaff(t)$ and $\theta(z_t, \Omega_t)$ are (up to a constant) powers of each other. We are interested in the case where this is a linear relation such that one actually has

$$Pfaff(t) = const. \cdot \theta(z_t, \Omega_t) \tag{6.23}$$

The theta divisor, when considered *vertically*, is described *locally* by a determinantal expression of rank $h^0(c, \widetilde{\mathcal{L}}|_{c_t})$ at $z(t)$: for this recall that, by a theorem of Kempf (cf. (B.10)), in our case of $d = g-1$ for $\mathcal{L} = K_c^{1/2} \otimes \mathcal{F}_t$, the codimension one analytic subvariety $\Theta_{-\kappa} = W_{g-1}$ can be described near $z = \mu(D)$ by an equation $\det f_{ij} = 0$ where f_{ij} is an $h \times h$ matrix of functions holomorphic at z where $h = h^0(c, \mathcal{O}_c(D))$ with specific linear terms. This refines (6.22) as thereby (when considered *vertically*, i.e. for fixed Ω) the divisor $\Theta_{(-\kappa)}$ is described *locally* by a hypersurface of degree h

$$ord_{z(\Omega_t)}\, \theta(\cdot, \Omega_t) = h^0(c_t, \widetilde{\mathcal{L}}|_{c_t}) \tag{6.24}$$

A Moduli spaces of SU(n) bundles

The moduli space of semistable SU(n)-bundles V_F on an elliptic curve F (so V_F decomposes as a sum of flat line bundles $L_i = \mathcal{O}_F(q_i - p_0)$ with $\sum q_i = 0$ in the group law) is a projective space, encoding the q_i as vanishing divisor of a meromorphic function w

$$\mathcal{M}_F(V_F) \;\cong\; \mathbf{P}H^0(F, \mathcal{O}_F(np_0)) \;\cong\; \mathbf{P^{n-1}} \tag{A.1}$$

from (in affine coordinates of $z = 1$; this is for n odd, otherwise the last power is $x^{n/2}$)

$$w = a_0 + a_2 x + a_3 y + \cdots + a_n x^{\frac{n-3}{2}} y = 0 \tag{A.2}$$

Now one proceeds [1] to the Calabi-Yau space X, elliptically fibered over the base surface B with fibre F and section σ. Using the line bundle $\mathcal{T} = K_B^{-1}$ over B one has the Weierstrass representation[27] of X over B (with the zero point $p = (0, 1, 0)$)

$$zy^2 \;=\; 4x^3 - g_2 x z^2 - g_3 z^3 \tag{A.3}$$

This represents X as a hypersurface in a fourfold W which is $\mathbf{P^2}$-fibered over B

$$W \;=\; \mathbf{P}(\mathcal{T}^2 \oplus \mathcal{T}^3 \oplus \mathcal{O}) \tag{A.4}$$

(with x, y, z as homogeneous coordinates). A polynomial representation of $\mathcal{E} = \pi^{-1}(b)$ arises just by restriction of the Weierstrass fibration of X over B to that of $\mathcal{E}$ over b.

An SU(n)-bundle over X (which is fibrewise suitably generic) determines a section of a $\mathbf{P^{n-1}}$-bundle $\mathcal{W}$ over B (a relative projective space over B) and has the $a_i \in H^0(B, \mathcal{T}^{-i})$ as homogeneous coordinates (more precisely $a_i \in H^0(B, \mathcal{O}_{\mathcal{W}}(1) \otimes \mathcal{T}^{-i})$) on the total space

$$\mathcal{W} \;=\; \mathbf{P}\Big(\mathcal{O}_B \oplus \bigoplus_{k=2}^{n} \mathcal{T}^{-k}\Big) \tag{A.5}$$

This gives for the moduli space of fibrewise semistable bundles a map

$$\mathcal{M}_X(V_X) \;\longrightarrow\; \Gamma(B, \mathcal{W}) \;=\; \mathbf{P}H^0\Big(B, \mathcal{O}_B \oplus \bigoplus_{k=2}^{n} \mathcal{T}^{-k}\Big) \tag{A.6}$$

For $s \in \Gamma(B, \mathcal{W})$ a further globalization datum is given by the twist line bundle $\mathcal{M} := s^*(\mathcal{O}_{\mathcal{W}}(1))$ over B (of $c_1(\mathcal{M}) = \eta$, say). Under s the a_i pullback to sections in $a_i \in H^0(B, \mathcal{M} \otimes \mathcal{T}^{-i})$. The a_i give via $[(a_0, \ldots, a_n)]$ the section in (A.6), cf. [1].

A.1 The spectral surface and the spectral curve

The spectral surface C is given by (A.2) with $w \in H^0(X, \mathcal{O}(\sigma)^n \otimes \pi^*\mathcal{M})$ where

$$\begin{array}{lll}
\underline{n=2} & w \;=\; a_0 z + a_2 x & \\
\underline{n=3} & w \;=\; a_0 z + a_2 x + a_3 y & (= C_r z + B_{r-2} x + A_{r-3} y\,) \\
\underline{n=4} & w \;=\; a_0 z^2 + a_2 x z + a_3 y z + a_4 x^2 & \\
\underline{n=5} & w \;=\; a_0 z^2 + a_2 x z + a_3 y z + a_4 x^2 + a_5 x y & \\
\underline{n=6} & w \;=\; a_0 z^3 + a_2 x z^2 + a_3 y z^2 + a_4 x^2 z + a_5 x y z + a_6 x^3 &
\end{array} \tag{A.7}$$

[27] Where $x, y, z \in H^0(X, \mathcal{O}_X(3\sigma) \otimes \mathcal{T}^i)$ with $i = 2, 3, 0$ and $g_2, g_3 \in H^0(X, \mathcal{T}^i)$ with $i = 4, 6$.

or in general (where $0 \le i, j$ and $j \le 1$; w has degree $[\frac{n}{2}] = \begin{cases} \frac{n}{2} & n\,\text{even} \\ \frac{n-1}{2} & n\,\text{odd} \end{cases}$ in x, y, z)

$$\begin{aligned} w &= \sum_{\substack{m=0 \\ 2i+3j=m}}^{n} a_m x^i y^j z^{[\frac{n}{2}]-(i+j)} \\ &= a_0 z^{[\frac{n}{2}]} + a_2 x z^{[\frac{n}{2}]-1} + a_3 y z^{[\frac{n}{2}]-1} + \ldots \\ &\quad + \begin{cases} a_{n-2} x^{[\frac{n}{2}]-1} z + a_{n-1} x^{[\frac{n}{2}]-2} yz \ + \ a_n\, x^{[\frac{n}{2}]} & n\,\text{even} \\ a_{n-3} x^{[\frac{n}{2}]-1} z + a_{n-2} x^{[\frac{n}{2}]-2} yz \ + a_{n-1} x^{[\frac{n}{2}]} + a_n x^{[\frac{n}{2}]-1} y & n\,\text{odd} \end{cases} \end{aligned} \tag{A.8}$$

For the elliptic curve $F \subset \mathbf{P^2}$ considered in (A.3), the divisor $(z) = l \subset \mathbf{P^2}$ becomes $(z)|_F = 3p_0$ on F where $p_0 = (0, 1, 0)$. To encode n points on F (for n even, say) one chooses a homogeneous polynomial $w^{(hom)}_{n/2}(x, y, z)$ of degree $n/2$. From its $3n/2$ zeroes on F only n, say q_i, carry information as $n/2$ of them are always at p_0: the rewriting $w^{(hom)}_{n/2}(x, y, z) = z^{n/2} w^{\text{aff}}_{n/2}(x/z, y/z)$ shows $3n/2$ zeroes at p_0, n poles at p_0 and n zeroes at the q_i (x/z and y/z have a pole at p_0 of order $-(1-3) = 2$ and $-(0-3) = 3$, resp.).

This gives on F for the divisor of $w^{(hom)} = w$ resp. for the divisor of zeroes of w^{aff}

$$(w|_F) = \frac{n}{2}\sigma + \sum q_i \,, \qquad (w^{\text{aff}}|_F)_0 = \sum q_i \tag{A.9}$$

Restricting from the elliptic threefold $X \subset W$ (where the fourfold W is the $\mathbf{P^2}$ bundle of Weierstrass coordinates over B) to the elliptic surface $\mathcal{E}$ one gets again (A.3) where now g_2 and g_3 are homogeneous polynomials of degree 4 and 6, respectively, on $b = \mathbf{P^1}$. Similarly one gets equation (A.2) for $c \subset \mathcal{E} \subset \mathcal{W}_b$ (inside the threefold $\mathcal{W}_b$ given by the $\mathbf{P^2}$ bundle over b) where $\mathcal{T}|_b = \mathcal{O}_b(\chi)$ with $x, y, z \in H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s + i\chi F))$ for $i = 2, 3, 0$ and $a_i|_b \in H^0(b, \mathcal{O}_b(r - i\chi))$ where $\mathcal{M}|_b = \mathcal{O}_b(r)$ and $r := \eta \cdot b$. The coefficients of the homogeneous polynomials a_i can be considered as moduli of external motions of c in $\mathcal{E}$. We set $\chi = 1$ as this is the case of b isolated (cf. section 3.1 of [9]).

A.2 The dimension of the moduli space

The moduli decomposition in (3.8) can also be seen from the action of the canonical involution τ (inversion on the fibers,trivially on the base; it sends y to $-y$ in Weierstrass coordinates). The base and (continuous) fibre moduli constitute the even and odd parts, respectively, cf. [8]. Considering the number n_e/n_o of τ-even/odd bundle moduli one has then $n_e - n_o = h^{2,0}(C) - h^{1,0}(C)$, cf. [8]. Considering [1] the τ-equivariant index $\chi_\tau(Z, ad\, V) = \sum_{i=0}^{3}(-1)^i\, \mathrm{Tr}_{H^i(X, ad\, V)}\tau$ (as the ordinary index $\chi(Z, ad\, V)$ vanishes by Serre duality) one gets (using the projector $(1+\tau)/2$ such that $I = -\sum_i (-1)^i h^i_{\text{even}}$)

$$I = -\frac{1}{2}\chi_\tau(Z, ad\, V) = -\sum_{i=0}^{3}(-1)^i\, \mathrm{Tr}_{H^i(X, ad\, V)} \frac{1+\tau}{2} = n_e - n_o$$

(for trivial unbroken gauge group, say, such that h^0_{even} and $h^3_{\text{even}} = h^0_{\text{odd}}$ are zero; note that Serre duality interchanges the even and odd subspaces of the respective cohomologies, the canonical bundle being odd). This gives [1]

$$I = rk - \sum_{i=1,2} \int_{U_i} c_2(V) \;=\; n - 1 - 4\omega_{\gamma=0} + \eta c_1 \tag{A.10}$$

where the U_i are the two fixed point sets σ and $\{y = 0\}$. Alternatively one gets [8]

$$\begin{aligned} I + 1 \;&=\; \chi\big(X, \mathcal{O}(C)\big) \;=\; \frac{1}{12}\big(c_2(C) + c_1^2(C)\big)C \;=\; \frac{1}{12}\big(c_2(X)C + 2C^3\big) \\ &= n + \frac{n^3 - n}{6}c_1^2 + \frac{n}{2}\eta(\eta - nc_1) + \eta c_1 \end{aligned} \tag{A.11}$$

(with $c_1 := c_1(B)$). In our case of $h^{1,0}(C) = 0$ one gets a formula for the number $h^{2,0}(C)$ of external deformations of C in X. Alternatively one can compute this[28] from the degrees of freedom in the coefficient functions a_i in the equation for the spectral cover (taking into account the possibility of an overall scaling of the defining equation) (cf. also (3.9))

$$\sharp \text{ of parameters in the } a_i = \sum_{\substack{i=0\\ i\neq 1}}^{n} \chi(B, \mathcal{M}\otimes\mathcal{T}^{-i}) - 1 = \sum_{\substack{i=0\\ i\neq 1}}^{n} h^0(B, \mathcal{O}_B(\eta - ic_1)) - 1 \tag{A.12}$$

A.3 Spectral cover bundles

As a further datum describing V beyond the surface C, which encodes V just fiberwise, one has a line bundle L over C with $V = p_*(p_C^* L \otimes \mathcal{P})$ where $p(= p_X)$ and p_C are projections to the respective factors in $X \times_B C$ and $\mathcal{P}$ the suitable Poincare line bundle [1]. L arises in the simplest case as a restriction $L = \underline{L}|_C$ to C of a line bundle $\underline{L}$ on X.

The line bundle L is, for C ample, encoded just by a half-integral number λ stemming from a twist class γ. This occurs as $c_1(V) = \pi_*\big(c_1(L) + \frac{c_1(C)-c_1}{2}\big) = 0$ implies

$$c_1(L) \;=\; -\frac{1}{2}\big(c_1(C) - \pi_{C*}c_1\big) + \gamma \;=\; \frac{n\sigma + \eta + c_1}{2}|_C + \gamma \tag{A.13}$$

(we omit the obvious pullbacks for the integral (1,1)-classes η and $c_1 = c_1(B)$ on B) where γ denotes the only generally given class in $ker\, \pi_{C*} : H^{1,1}(C) \to H^{1,1}(B)$

$$\gamma \;=\; \Gamma|_C \quad \text{with} \quad \Gamma \;=\; \lambda\,\big(n\sigma - (\eta - nc_1)\big) \tag{A.14}$$

(here Γ is an element of $H^{1,1}(X)$; as in [9] we will always assume $\lambda > 1/2$).

This gives precise integrality conditions for $\lambda \in \frac{1}{2}\mathbf{Z}$: if n is odd, then one needs actually $\lambda \in \frac{1}{2} + \mathbf{Z}$; if n is even, then $\lambda \in \frac{1}{2} + \mathbf{Z}$ needs $c_1 \equiv 0 \bmod 2$ and $\lambda \in \mathbf{Z}$ needs $\eta \equiv c_1 \bmod 2$. The nontriviality of this parameter is crucial to get chiral matter [7].

Under our assumption $h^{1,0}(C) = 0$ line bundles on C are characterised by their Chern classes. Therefore one can define line bundles $\underline{\mathcal{G}}$ and $\mathcal{G}$ on X and C, respectively, by

$$c_1(\underline{\mathcal{G}}) \;=\; \Gamma, \quad c_1(\mathcal{G}) \;=\; \gamma \tag{A.15}$$

[28] Assuming that $\eta - nc_1$ is not only effective but even positive (cf. also [10]).

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(when we want to make the λ-dependence explicit we denote these by $\underline{\mathcal{G}}_\lambda$ and $\mathcal{G}_\lambda$). Further we will use the corresponding divisor classes (modulo linear equivalence) $\underline{G}$ and G with

$$\underline{\mathcal{G}} = \mathcal{O}_X(\underline{G}) \ , \ \mathcal{G} = \mathcal{O}_C(G) \tag{A.16}$$

i.e. one has $\underline{G} = \lambda(n\sigma - \pi^*(M + nK_B))$ and, for example, $G|_c = \lambda(ns - (r - n\chi)F)|_c$.

Note that all these considerations of $\underline{\mathcal{G}}$ and $\mathcal{G}$ apply strictly only formally as the corresponding Chern classes will, taken alone for themselves, be only half-integral in general; only the full combination in (A.13) will be integral and define a proper line bundle. Similar remarks apply to decompositions written below.

Explicitely one finds for the various incarnations of the spectral line bundle

$$\underline{L} = \left(\mathcal{O}_X(\sigma)^n \otimes \pi^*\mathcal{M} \otimes \pi^* K_B^{-1}\right)^{1/2} \otimes \underline{\mathcal{G}} = \mathcal{O}_X\left(\frac{C + \pi^* K_B^{-1}}{2} + \underline{G}\right) \tag{A.17}$$

$$L = K_C^{1/2} \otimes \pi_C^* K_B^{-1/2} \otimes \mathcal{G} \tag{A.18}$$

The spectral data along the elliptic surface $\mathcal{E}$. We define also a restriction $l := \underline{L}|_{\mathcal{E}}$ to $\mathcal{E}$. So one has the inclusions of line bundles

$$\begin{array}{ccc} L & \hookrightarrow & \underline{L} \\ \downarrow & & \downarrow \\ C & \hookrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} l|_c & \hookrightarrow & l \\ \downarrow & & \downarrow \\ c & \hookrightarrow & \mathcal{E} \end{array} \tag{A.19}$$

The crucial fact is that one has for spectral bundles

$$V|_B = \pi_{C*}L \qquad \text{such that} \qquad V|_b = \pi_{c*}\, l|_c \tag{A.20}$$

Let us define the following expression related to the restriction to $\mathcal{E}$

$$r := \eta \cdot b \in \mathbf{Z} \tag{A.21}$$

So $C = n\sigma + \eta$ gives $c = ns + rF$. One has by $\eta \geq nc_1$ that $r \geq n\,\chi$. For C ample one gets even $r > n$, cf. equ. (3.12) of [9]. $Pfaff \not\equiv 0$ needs $r \geq \frac{n\lambda + 1/2}{\lambda - 1/2}$ by (4.28) of [9], so in the SU(3) case one has even $r \geq 5$ for $\lambda = 3/2$ (while just $r \geq 4$ for $\lambda > 3/2$).

One has with $\Gamma|_{\mathcal{E}} = \lambda\big(ns - (r - n\chi)F\big)$ and $\mathcal{T}|_b = K_B^{-1}|_b = \mathcal{O}_b(\chi)$ that[29] (with $l|_c = L|_c$)

$$\begin{aligned} l &= \underline{L}|_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}\left(\frac{c + \pi_{\mathcal{E}}^{-1}\mathcal{O}_b(\chi)}{2} + \underline{G}|_{\mathcal{E}}\right) = \mathcal{O}_{\mathcal{E}}\left(\frac{ns + (r+\chi)F}{2}\right) \otimes \underline{\mathcal{G}}|_{\mathcal{E}} \\ &\Longrightarrow l(-F) = \big(K_{\mathcal{E}} \otimes \mathcal{O}_{\mathcal{E}}(c)\big)^{1/2} \otimes \mathcal{O}_{\mathcal{E}}\big(ns - (r - n\chi)F\big)^{\lambda} \end{aligned} \tag{A.22}$$

$$\begin{aligned} l|_c &= \mathcal{O}_c\left(\left.\frac{ns + (r+\chi)F}{2}\right|_c\right) \otimes \mathcal{F} = \left.\left(K_C^{1/2} \otimes \pi_C^* K_B^{-1/2}\right)\right|_c \otimes \mathcal{F} = K_c^{1/2} \otimes \pi_c^* K_b^{-1/2} \otimes \mathcal{F} \\ &\Longrightarrow \mathcal{L} = l(-F)|_c = K_c^{1/2} \otimes \mathcal{F} = \mathcal{O}_{\mathcal{E}}(\alpha s + \beta F)|_c \end{aligned} \tag{A.23}$$

[29] The equality to the second line in (A.23) follows from $l^2|_c \otimes \mathcal{F}^{-2} = \mathcal{O}_{\mathcal{E}}\big((ns + rF) + \chi F\big)|_c = \mathcal{O}_{\mathcal{E}}\big((ns + rF - kF) + 2F\big)|_c = K_c \otimes \pi_c^* K_b^{-1}$ using $K_c = (c + K_{\mathcal{E}})c$ and $K_{\mathcal{E}} = -kF = (\chi - 2)F$.

where

$$\alpha = n\Big(\lambda + \frac{1}{2}\Big), \qquad \beta = \Big(n\lambda + \frac{1}{2}\Big)\chi - \Big(\lambda - \frac{1}{2}\Big)r - 1 \tag{A.24}$$

Note that our standing assumption $\chi = 1$ gives these precise integrality conditions for $\lambda \in \frac{1}{2}\mathbf{Z}$: for n odd one has $\lambda \in \frac{1}{2} + \mathbf{Z}$, for n even one has $\lambda \in \mathbf{Z}$ and r odd.

Here we have also introduced the line bundle on c given by the restriction[30]

$$\mathcal{F} = \mathcal{G}|_c = \mathcal{O}_c\big(G|_c\big) \tag{A.25}$$

One notes that the line bundle $\mathcal{F}$ is flat as

$$\big(ns - (r - n\chi)F\big)(ns + rF) = 0 \Longrightarrow \deg G|_c = 0 \tag{A.26}$$

The question whether b contributes to W depends only on $V|_b$. The class γ does not occur for a spectral bundle $V_{\mathcal{E}}$ over b where $\pi_{c*} : H^{1,1}(c) \longrightarrow H^{1,1}(b)$ is injective, cf. [1]. This is in accord with the fact that $c_2(V|_{\mathcal{E}}) = r$ sees only the γ-free part of $c_2(V) = \eta\sigma + \omega$ and not the γ-dependent pullback-class ω (there are some further consequences along these lines not pursued here).

A.4 An auxiliary locus $(R \subset \overline{R})$

To elucidate some of our examples of SU(3) bundles we give a technical discussion of an auxiliary locus: one has a second explicit locus of codimension $n-2$ ($R_i := Res(a_i, a_n)$)

$$(\mathcal{R} \subset)\;\; R := \bigcap_{i=2,\dots,n-1} (R_i) \tag{A.27}$$

For $r = n+1$ (where $\deg a_n = 1$ and $j = 1$ in (4.33)) one has $\mathcal{R} = \mathcal{R}^1 = R$ (cf. below).

In our consideration above we compared two loci: the structurally defined locus Σ and its sublocus $\mathcal{R}$ which is defined by explicit equations. In direct analogy to the discussion in (4.32) and (4.33) let us define the following loci

$$(\Sigma \;\subset) \qquad \Sigma^1 := \big\{t \in \mathcal{M}_{\mathcal{E}}(c) \,\big|\, (ns - F)|_{c_t} \sim \text{effective}\big\} \tag{A.28}$$

$$\cup \qquad\qquad \cup$$

$$(\mathcal{R} \;\subset) \qquad \mathcal{R}^1 := \big\{t \in \mathcal{M}_{\mathcal{E}}(c) \,\big|\, (ns - F)|_{c_t} = \text{effective}\big\} \tag{A.29}$$

so $t \in \mathcal{R}^1$ just if $F_{u_j}|_c = \{np_0, u_j\}$ for some j. Concerning the locus Σ^1 one has

$$\Sigma^1 = \begin{cases} \mathcal{M}_{\mathcal{E}}(c) & \text{for } \; r \geq n\frac{n+1}{2} \\ \Sigma & \text{for } \; r = n+1 \end{cases} \tag{A.30}$$

(if $r = n+1$ is attainable, cf. after (A.21)). The second relation is obvious; the first one follows from the R-R computation $h^0(c_t, \mathcal{O}_{\mathcal{E}}(ns - F)|_{c_t}) = n(r - n - 1) - n(r - \frac{n+1}{2}) + h^1(c_t, \mathcal{O}_{\mathcal{E}}(ns - F)|_{c_t}) = h^0(c_t, \mathcal{O}_{\mathcal{E}}(rF)|_{c_t}) - n\frac{n+1}{2}$ together with $h^0(c_t, \mathcal{O}_{\mathcal{E}}(rF)|_{c_t}) \geq r + 1$ from the long exact sequence associated with the restriction sequence from $\mathcal{E}$ to c_t.

[30] The line bundle $\mathcal{F}$ *on* c will exist without the proviso concerning half-integrality because $K_c^{1/2}$ will exist (although with a $Pic^0_{(2)}(c)$-ambiguity) while $\mathcal{G}|_c$ and $G|_c$ still must be read with this proviso.

SU(3) bundles. In view of the examples mentioned we study more closely the case $n = 3$. Here one gets (where we use already that that $\mathcal{R} = \Sigma$ for $n = 3$, cf. (4.48); for $r \geq 6$ use R-R)

$$\Sigma^1 = \begin{cases} \mathcal{M}_{\mathcal{E}}(c) & \text{for } \; r \geq 6 \\ \mathcal{R}^1 = R & \text{for } \; r = 5 \\ \Sigma = \mathcal{R} = \mathcal{R}^1 = R & \text{for } \; r = 4 \end{cases} \tag{A.31}$$

(where $r = 4$ needs $\lambda > 3/2$, cf. after (A.21)). Here for $r = 5$ the non-triviality of $H^0(c_t, \mathcal{O}_{\mathcal{E}}(3s - F)|_{c_t})$ is controlled by $\det \bar{\iota}_1$ where $\bar{\iota}_1$ is the map in the long exact sequence

$$0 \longrightarrow H^0\big(c_t, \mathcal{O}_{\mathcal{E}}(3s - F)|_{c_t}\big) \longrightarrow H^1\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(-(r+1)F)|_{c_t}\big) \xrightarrow{\bar{\iota}_1} H^1\big(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(3s - F)|_{c_t}\big)$$

Here the second term is the five-dimensional space $H^1(b, \mathcal{O}_b(-(r+1))) \cong H^0(b, \mathcal{O}_b(r - 1))^*$ and the third term is the five-dimensional space $H^1(b, \mathcal{O}_b(-1) \oplus \bigoplus_{i=2}^3 \mathcal{O}_b(-1-i)) \cong H^0(b, \mathcal{O}_b(1) \oplus \mathcal{O}_b(2))^*$. Now, here $R = (R_2)$ and the concrete expression for $\det \bar{\iota}_1$ from equ. (G.13) in [9] is just $R_2 = Res(a_2, a_3) = Res(B_3, A_2)$ which controlles $\mathcal{R}^1$.

B Riemann theta function and Jacobian fibration

B.1 Preliminaries concerning abelian varieties

Recall that a complex torus $M = V/\Lambda$ (with $V \cong \mathbf{C^g}$ and $\Lambda \cong \mathbf{Z^{2g}}$ a discrete lattice) is an abelian variety, i.e. an (projective) algebraic variety (allowing a projective embedding), exactly if it admits a Kähler form H in the rational cohomology (coming from the basis dx_i dual to the integral Λ-basis of topological cycles, cf. below), a so-called Hodge form (by averaging over the compact space M one can always consider just invariant forms of this type); the choice of cohomology class of H, or the corresponding datum of an ample line bundle L on M, is then called a polarization.

There are two bases on $H^1(M, \mathbf{C})$: if $z_i, i = 1, \ldots, g$ are Euclidean coordinates one gets the corresponding 1-forms dz_i and $d\bar{z}_i$; on the other hand if $\lambda_i, i = 1, \ldots, 2g$ are elements of an integral basis of Λ they build also a basis for the real vector space V; this gives dual real coordinates $x_i, i = 1, \ldots, 2g$ on V and corresponding 1-forms $dx_i, i = 1, \ldots, 2g$ with $\int_{\lambda_j} dx_i = \delta_{ij}$ for $\lambda_j \in \Lambda \cong H_1(M, \mathbf{Z})$. If $e_i (i = 1, \ldots, g)$ is a complex basis of V the $g \times 2g$ period matrix Ω' effects via $dz_i = \Omega'_{ij} dx_j$ (and the conjugate for $d\bar{z}_i$) the change between the two mentioned bases. In the topological basis one gets for the integral (invariant) 2-form H (where the δ_α are integers with $\delta_\alpha | \delta_{\alpha+1}$)

$$H = \sum \delta_\alpha \, dx_\alpha \wedge dx_{g+\alpha} \tag{B.1}$$

H is called a principal polarization if the elementary divisors δ_α are all equal to 1. (M, L) is then called a principally polarised abelian variety (p.p.a.v.). Equivalently this means that the corresponding line bundle L has an essentially unique non-trivial section, i.e. $h^0(M, L) = 1$. Precisely then the period matrix can (using $e_i = \delta_i^{-1} \lambda_i$) be normalized to

$$\Omega' = (I, \Omega) \tag{B.2}$$

where Ω is an element of the Siegel upper half plane $\mathcal{H}_g$ (symmetric matrices of positive definite imaginary part). Conversely, given a point $\Omega \in \mathcal{H}_g$, one gets the complex torus $M_\Omega = \mathbf{C}^g/\Lambda_\Omega$ from the lattice Λ_Ω generated by the columns of (I, Ω).

A line bundle L on M arises as quotient of $V \times \mathbf{C}$ under $(z, w) \sim (z+\lambda, e_\lambda(z)w)$ with a system of non-vanishing holomorphic functions $(e_\lambda)_{\lambda\in\Lambda}$ with $e_{\lambda_2}(z+\lambda_1)e_{\lambda_1}(z) = e_{\lambda_1+\lambda_2}(z)$. For z_i coordinates on V dual to a basis e_i and L defined by $e_{\lambda_\alpha} = 1, e_{\lambda_{g+\alpha}} = \exp(-2\pi i z_\alpha)$ sections $\tilde{\theta}$ are given by functions θ changing by the corresponding multiplier under λ_α-shifts in z. Then $c_1(L) = [H]$ determines L up to translation and the translates $\tau_\mu^* L$ are just the line bundles with the same multipliers, except that the $e_{\lambda_{g+\alpha}}$ are muliplied by constants $c_\alpha = \exp(-2\pi i \mu_\alpha)$: for $\mu = \frac{1}{2}\sum \Omega_{\alpha\alpha} e_\alpha$, say, one has the transformation laws

$$\theta(z + \lambda_\alpha) = \theta(z)\,, \qquad \theta(z + \lambda_{g+\alpha}) = \exp\left(-2\pi i\left(\frac{1}{2}\Omega_{\alpha\alpha} + z_\alpha\right)\right)\theta(z) \tag{B.3}$$

This characterises θ (up to a factor): one has (we consider principal polarizations H) $H^0(M, L) = \mathbf{C}\theta$ where (here θ is considered as a function on $\mathbf{C}^\mathbf{g} \times \mathcal{H}_g$)

$$\theta(z, \Omega) = \sum_{l\in\mathbf{Z}^\mathbf{g}} \exp\left\{2\pi i\left(\frac{1}{2}\langle l, \Omega l\rangle + \langle l, z\rangle\right)\right\} \tag{B.4}$$

(with $\langle\cdot,\cdot\rangle$ the canonical scalar product on $\mathbf{C}^\mathbf{g}$). $\theta(\cdot, \Omega)$ is the Riemann theta-function (an even holomorphic function) of divisor $\Theta = \Theta_\Omega$ (the quasi-periodicity of θ, considered as a function, shows that its zeroes are well-defined on M_Ω). We denote a translate by $\Theta_\lambda := \Theta + \lambda$. If Θ is the zero-divisor $(\tilde{\theta})$ of a section $\tilde{\theta}$ of L one has $\Theta = c_1(L)$ such that

$$L \;=\; \mathcal{O}_M(\Theta) \tag{B.5}$$

Conversely, any principally polarised abelian variety (M, L) is of the form $(M_\Omega, \Theta_\Omega)$. The moduli space[31] $\mathcal{A}_g$ of p.p.a.v.'s is a quasi-projective variety of dimension $g(g + 1)/2$

$$\mathcal{A}_g \;=\; \mathcal{H}_g/\mathrm{Sp}(2g, \mathbf{Z}) \tag{B.6}$$

B.2 The Jacobian and Jacobi's inversion of Abel's theorem

For c a curve Ω' gives the integrals of a basis $(\omega_\alpha) \in H^0(c, \Omega^1)$ of holomorphic differentials over a symplectic basis of topological cycles in $H_1(c, \mathbf{Z})$ and the Jacobian $Jac(c) = H^0(c, \Omega^1)^*/H_1(c, \mathbf{Z})$ is a prinipally polarized abelian variety (essentially by Poincare duality; the polarization can be identified with the intersection form of c).

The moduli space of curves of genus g is a quasi-projective variety $\mathcal{M}_g$ of dimension $3g-3$. Considering c together with a symplectic basis of $H_1(c, \mathbf{Z})$ gives a topological cover $\tilde{\mathcal{M}}_g$ as moduli space and the period map (giving Ω) defines a holomorphic map

$$\tilde{\mathcal{M}}_g \xrightarrow{\tilde{P}} \mathcal{H}_g \tag{B.7}$$

Let us denote its image, the jacobian locus, by $\mathcal{J}_g$. $\tilde{P}$ clearly descends to a map

$$\mathcal{M}_g \xrightarrow{P} \mathcal{A}_g \tag{B.8}$$

[31] (M, L) and (M', L') are called isomorphic if an isomorphism $f : M \to M'$ exists with $f^*L' = L$.

Here the map P is injective by Torelli's theorem, thus providing the identification $\mathcal{M}_g \cong \mathcal{J}_g/\mathrm{Sp}(2g, \mathbf{Z})$. The curve c can thus be recovered from $(Jac(c), \Theta_c)$.

θ lives naturally on the universal Jacobian bundle[32] $\underline{Jac}$, the total space of the fibration

$$\begin{array}{c} \underline{Jac} \\ p \downarrow \; Jac(\cdot) \\ \mathcal{M}_g \end{array} \tag{B.9}$$

Let $\mu : c \longrightarrow Jac(c)$ be given by the integrals of the ω_i from a fixed point $p_0 \in c$ to $p \in c$. One has $\mu(c) \cdot \Theta = g$, the intersection consisting of g points (counting multiplicities) if not $\mu(c) \subset \Theta$. It was Jacobi's insight that this realizes an inversion to Abel's Theorem. The latter was the assertion that the map $D \to \mu(D) = (\sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_i)_{i=1,\ldots,g}$ for divisors $D = \sum(p_\lambda - q_\lambda)$ of degree zero becomes injective when working modulo linear equivalence, i.e. mediates an injection of the Pic_0 group to the Jacobian. Jacobi's inversion stated (fixing a reference point $p_0 \in c$) that for $\lambda \in Jac(c)$ a (generically uniquely determined) effective divisor $D = \sum_{i=1}^g p_i \in c^{(g)} := Sym^g c$ exists with $\sum(p_i - p_0)$ being mapped to $\lambda - \kappa$, explicitely $\{p_i\} = \mu(c) \cap \Theta_\lambda$ if not $\mu(c) \subset \Theta_\lambda$ (or equivalently if not $\lambda = \kappa + \mu(D_g)$ with $h^0(c, D_g) > 1$ where $D \in c^{(g)}$); so one has $W_g := \mu(Sym^g c) = Jac(c)$.

Riemann's theorem asserts that $\mathrm{mult}_z\, \Theta = h^0(c, \mathcal{O}_c(D_z))$ where codim $\Theta_{\mathrm{sing}} = 3$ if c is not hyperelliptic (for c hyperelliptic it is 2). More precisely the codimension one analytic subvariety $\Theta_{-\kappa} = W_{g-1}$ can be described near $z = \mu(D)$ by an equation $\det f_{ij} = 0$ where f_{ij} is an $h \times h$ matrix of functions holomorphic at z where $h = h^0(c, \mathcal{O}_c(D))$. For this recall for effective divisors D the map $\mu: \; c^{(d)} \; \to \; Jac(c)$ with $d = \deg\, D$, cf. (6.3). Here one identifies the following kernel and cokernel of the induced differential map $d\mu$ on tangent spaces at D and $z = \mu(D)$, respectively

$$0 \longrightarrow T_D|D| \longrightarrow T_D c^{(d)} \xrightarrow{d\mu} T_{z=\mu(D)} Jac(c) \longrightarrow H^0(c, K_c - D)^* \longrightarrow 0 \tag{B.10}$$

Taking dimensions here gives Riemann-Roch: $\dim |D| - d + g - h^0(c, K_c - D) = 0$. For a basis (f_i) of $\Phi(D) := \{\text{meromorphic } f : c \to \mathbf{C} \,|\, (f) + D \geq 0\}$ the f_i span the vector space of translations of the affine space $|D|$: an element $D' \in |D|$ is given as $D + (f')$ for an $f' = \sum a_i' f_i$. Correspondingly then, for a basis (ω_j) of $H^0(c, K_c - D)$, the $f' \omega_j$ span $H^0(c, K_c - D')$ (the dual of the cokernel of $d\mu$ at D') i.e. their common vanishing characterizes $im\, d\mu$ in $T_{z'=\mu(D')} Jac(c)$; in other words, the $h \times (g - 1 - d + h)$ matrix of linear forms $(f_i\, \omega_j)$, acting on $T_{z=\mu(D)} Jac(c)$, has all $h \times h$ minors vanishing on the image in the tangent cone (Kempf proved that these equations are also sufficient). What was considered here infinitesimally can also be described locally, i.e. as local equations for $d\mu$ which have the $(f_i\, \omega_j)$ as linear parts.

B.3 Theta characteristics

Consider the set of spin bundles of c, i.e. the set of square roots of the canonical bundle[33]

$$\mathcal{S}p(c) = \big\{D \in (Div(c)/\sim) \big| 2D \sim K_c\big\} \;\cong\; \big\{\mathcal{O}_c(D) \in Pic(c) \big| \mathcal{O}_c(D)^2 \cong K_c\big\} \tag{B.11}$$

[32] Here, and when including spin structures below, we will have no need to include possible subtleties, for example stemming from curves with automorphisms, explicitly in the discussion ($\underline{Jac}$ is used naively).

[33] Using for the additive divisor class and the associated multiplicative line bundle K_c the same symbol.

This set of socalled theta characteristics (or different spin structures) is a $Pic_0^2(c)$-torsor, i.e. its elements are just rotated through by this group of square roots of the trivial line bundle (which themselves lie in the degree zero component $Pic_0(c)$ of $Pic(c)$ whereas the theta characteristics have degree $g-1$); the set of these 2-torsion points is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{2g}$ and has 2^{2g} elements; the notation $K_c^{1/2}$ has this inherent 2^{2g_c}-fold ambiguity.

Recall Riemann's theorem $W_{g-1} - \Delta = \Theta$ where $\Delta = \mu(D_0) - \mu((g-1)P_0)$. Sending a theta characteristic D to $W_{g-1} - \mu(D)$ establishes an isomorphism to the set of those translates Θ_e which are symmetric (which just comes down to $e \in Pic_0^2(c)$; here D_0 corresponds just to Θ itself). This sends $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a' \\ \frac{1}{2}b' \end{pmatrix} \in (\frac{1}{2}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$ to the zero locus of a shifted theta function (where $D = D_0 + d$ with $\mu(d) = \Omega a + b$)

$$\begin{aligned} \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega) &= \sum_{l \in \mathbf{Z^g}} \exp\left\{ 2\pi i \left(\frac{1}{2}\Big\langle l+a, \Omega(l+a) \Big\rangle + \Big\langle l+a, z+b \Big\rangle \right) \right\} \\ &= \exp\left\{ 2\pi i \left(\frac{1}{2}\Big\langle a, \Omega a \Big\rangle + \Big\langle a, z+b \Big\rangle \right) \right\} \theta(z + \Omega a + b, \Omega) \end{aligned} \tag{B.12}$$

(cf. (B.4)). $\mathcal{S}p(c)$ is divided into two sets $\mathcal{S}p_\pm(c)$ of $\frac{2^g(2^g \pm 1)}{2}$ even/odd structures

$$\mathcal{S}p(c) = \mathcal{S}p_+(c) \,\dot{\cup}\, \mathcal{S}p_-(c) \tag{B.13}$$

according to the parity of $mult_D\Theta$ or equivalently $h^0(c, \mathcal{O}_c(D)) \equiv a'^t \cdot b'(2)$. This parity stays constant in any family. We denote the 2^{2g}-section of 2-torsion points of $p : \underline{Jac} \longrightarrow \mathcal{M}_g$ in (B.9) by

$$Z = \Big\{ [z] \in Jac(c) \cong \mathbf{C^g}/\Lambda \,\Big|\, 2[z] = [0],\, \Omega_c \in \mathcal{M}_g \Big\} = Z_+ \,\dot{\cup}\, Z_- \tag{B.14}$$

where $[z = \frac{1}{2}\Omega a' + \frac{1}{2}b'] \in Z_\pm$ according to the parity of $a'^t \cdot b'$; so, this 2^{2g}-section Z decomposes in a $2^{g-1}(2^g+1)$-section Z_+ and a $2^{g-1}(2^g-1)$-section Z_-.

For example, a nonsingular odd (with $h^0(c, \mathcal{O}(D)) = 1$) theta characteristic $D \in Pic_{g-1}(c)$ (which always exists) corresponds to $\begin{pmatrix} a \\ b \end{pmatrix} \in (\frac{1}{2}\mathbf{Z})^{2g}/\mathbf{Z}^{2g} \cong Pic_2^0(c)$ where the shifted theta function has just a first order zero in z. On a generic curve a theta characteristic has actually $h^0(c, \mathcal{O}_c(D)) = 0$ or 1. The locus of curves in $\mathcal{M}_g$ (for $g \geq 3$) having an even theta characteristic with[34] non-vanishing $h^0(c, \mathcal{O}_c(D))$ is an irreducible divisor which we denote by $\mathcal{M}_g^1$

$$\mathcal{M}_g^1 = \big\{ c \in \mathcal{M}_g \,\big|\, \exists D \in \mathcal{S}p_+(c) \;\; s.t. \;\; h^0(c, \mathcal{O}_c(D)) \neq 0 \big\} \tag{B.15}$$

So one has the following decomposition of $Z \cap \Theta$ and projection of $Z_+ \cap \Theta$

$$\begin{array}{c} Z \cap \Theta = (Z_+ \cap \Theta) \;\dot{\cup}\; Z_- \\ \downarrow p \\ \mathcal{M}_g^1 \end{array} \tag{B.16}$$

[34] Or, equivalently, vanishing "theta-null", i.e. $\theta[{a \atop b}]([0], \Omega_c) = 0$ or $\theta(\Omega_c\, a + b, \Omega_c) = 0$; here the notion comes from "theta-nullwerte", that is theta zero-values (i.e. values at zero-argument in z).

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B.4 Some standard notation

We collect some mathematical standard notation and point to the place where the corresponding notion is considered in greater detail.

- $\theta(\cdot,\cdot)$ denotes the theta function, cf. appendix B.1.
- $Jac(c)$ denotes the Jacobian of the curve c, cf. appendix B.2.
- $Pic(c)$ denotes the group of line bundles over c; $Pic_m(c)$ denotes those line bundles which are of degree m; $Pic_0^k(c)$ denotes the k-torsion line bundles, i.e. those L for which $L^k \cong \mathcal{O}_c$ holds.
- $Sym^k c$ denotes the symmetric product of the point set c, i.e. all k-tuples of points of c.
- $\mathrm{Div}(c)$ denotes the group of divisors on c; $\mathrm{Div}_m(c)$ denotes those divisors which are of degree m; $\mathrm{Div}^{\mathrm{eff}}(c)$ denotes the set of effective divisors, so in particular $\mathrm{Div}_m^{\mathrm{eff}}(c) = Sym^m(c)$. Linear equivalence of divisors is indicated as usual by $\sim$, so $\mathrm{Div}_m(c)/\sim$ is isomorphic to $Pic_m(c)$.
- $\mu : \mathrm{Div}_m^{\mathrm{eff}}(c)/\sim \longrightarrow Jac(c)$ denotes the Jacobi map, cf. appendix B.2, and W_m its image; one has $W_{g-1} = \Theta + \mu(\frac{1}{2}K_c)$ according to Riemann's theorem (g is the genus of c).

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